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TUTORIAL ARTICLE

Remarks on Transitions Order-chaos Induced by the Shape of the Periodic Excitation in a Parametric Pendulum

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Abstract—We study a pendulum parametrically excited by nonharmonic perturbations. Instead of using a circular harmonic function to perturb the pendulum, we use a Jacobi elliptic function as a perturbation, which encompasses it as a limit. Melnikov analysis provides the general condition for the onset of homoclinic bifurcations, adding now the elliptic modulus as a new parameter. Using the elliptic modulus, which is responsible for the shape of the perturbation, as a control parameter, new transitions order-chaos may occur.

1. INTRODUCTION

The study of periodically excited nonlinear oscillators has been a topic of interest in the past few years. Usually a periodical excitation depending on one or several frequencies, acting either as an external perturbation to the system, or on the dynamical state variable in a parametric way, has been considered. Apart from the intrinsic dynamical properties of the response of the different oscillators to the excitations, one of the more striking properties under consideration has been the chaotic response. In this way current research has been mainly interested in finding out the regions of parameters, for which the responses of the nonlinear oscillators are chaotic. On the other hand, another topic of interest in recent years has been concerned with the suppression or inhibition of chaos. This has been done experimentally [1] and analytically, via the information which supplies the Melnikov method [2–6].

One of the ways to suppress chaos that has been proposed quite recently, deals with the use of parametric modulation. The effect of periodic perturbations on dynamical systems near the onset of a period-doubling bifurcation, considering near-resonant perturbations, has been studied in [7]. There it was found that near-resonant perturbations suppress the onset of subharmonic oscillations, which results in the suppression of period-doubling, or that the bifurcation point of an unperturbed system is shifted due to the presence of the small-resonant perturbation, in a way that stabilizes the system. Also Saravanan *et al.* [8] studied the parametrically modulated Lorenz system which can show hastening of chaos as well as stabilization of the attractor giving rise to a limit cycle, depending upon the

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amplitude of the modulation. The given theoretical explanation is that the modulation affects the threshold of the Hopf bifurcation and hence the threshold for the strange attractor is lowered.

Usually the perturbation that has been used in order to excite the nonlinear oscillator has been a harmonic function. However, a different way has been proposed recently, namely to excite the oscillator through the Jacobi elliptic functions [5, 6], which are the nonlinear generalization of the harmonic ones. This selection seems natural, since as harmonic functions are *natural* in a linear world, the Jacobi elliptic functions are perceived as *natural* in a nonlinear world.

In the present paper we consider the parametrically excited pendulum. This system has received considerable attention from many authors. The numerical studies of [9] show that strange attractor and period-doubling bifurcation cascades occur in both the dissipative and the conservative case. An experimental control of a chaotic parametrically driven pendulum has been reported in [10] and analytical studies can be found in [11]. However, all these studies consider only harmonic perturbations, and here we will consider nonharmonic perturbations. In fact, the perturbation we use is the nonlinear generalization of the harmonic excitation used in [11], but in such a way that we recover the same results as a limiting case. The existence of chaotic regions in parameter space due to homoclinic intersections are well known. The transitions from an ordered state to a chaotic one depends on the dissipation and on the frequency as well as the amplitude of the modulation.

We stress that the transitions which are reported here, correspond to a fairly new kind, which are induced by the modification of the elliptic modulus of the Jacobi elliptic function, representing the external perturbation. Even fixing the parameters in a chaotic state, we could eventually convert this chaotic state into an ordered one by simply modifying the elliptic modulus, used as a control parameter.

2. ONSET OF THE HOMOCLINIC BIFURCATIONS

We consider here the effect of periodic nonharmonic pulses, when they act as a parametric modulation on the pendulum equation. The parametric perturbation that we consider is the Jacobi cosine amplitude elliptic function, *cn*, of frequency ω and elliptic modulus k. The equations of motion for the parametrically forced pendulum with dissipation can be written as a system of first order differential equations:

$$\dot{x} = v, \qquad (1)$$
$$\dot{v} = -\varepsilon\beta v - (1 + \varepsilon A cn(\omega t,\, k))\sin x,$$

where $(x, v) \in S^1 \times R$, β, A, ω are positive constants and $0 < \varepsilon \leq 1$.

The application of the Melnikov method [12], allows us to ascertain the condition that the parameters A and β must satisfy in order that the homoclinic tangencies occur, and its corresponding chaotic response.

In order to calculate the onset of the homoclinic bifurcations, we must compute the Melnikov function for the parametrically excited pendulum in equation (1), which is to be considered as a planar Hamiltonian system, whose unperturbed Hamiltonian is

$$H = \frac{v^2}{2} - \cos x. \qquad (2)$$

For this purpose, the solutions of the unperturbed pendulum, $\varepsilon = 0$, must be known. The phase space of the pendulum is 2π-periodic with hyperbolic saddles in $(\pm\pi, 0)$ and an elliptic centre in $(0, 0)$. There are three different kind of orbits for the unperturbed pendulum: rotations, oscillations and separatrix motion. The rotations are the unbounded motions, the oscillations are the bounded ones and the separatrix motion correspond to an oscillation of infinite period. In fact, we can identify both hyperbolic saddles and consider a cylindrical phase space, and this allows us also to talk about homoclinic connections and homoclinic motions, which correspond to the separatrix motion, instead of the heteroclinic ones. The solutions for the oscillating orbits can be expressed in terms of the Jacobi elliptic functions as

$$(x(t), v(t)) = (2k\,cn(\omega t, k)\,dn(\omega t, k), 2k\,cn(\omega t, k)). \tag{3}$$

From them we can obtain the solutions for the homoclinic orbit, taking the limit $k \to 1$ for the elliptic modulus. In fact k is related to the energy or Hamiltonian of the unperturbed pendulum through the expression

$$H = 2k^2 - 1, \tag{4}$$

in such a way that it labels the different orbits. The homoclinic solutions are thus

$$(x_{sx}(t), v_{sx}(t)) = (2 \tanh \omega t \operatorname{sech} \omega t, 2 \operatorname{sech} \omega t). \tag{5}$$

The Melnikov function has to be evaluated for the homoclinic orbit and takes in our case the following form:

$$M(k, \omega; t_0) = \int_{-\infty}^{+\infty} v_{sx}(t - t_0)\{-A\,cn(\omega t, k) \sin(x(t - t_0)) - \beta v_{sx}(t - t_0)\}\,dt. \tag{6}$$

In order to calculate this integral, which involves the Jacobi elliptic function cn, we will consider its Fourier expansion, which is given by

$$cn(\omega t, k) = \frac{1}{k}\frac{\pi}{K}\sum_{n=0}^{n=\infty} \operatorname{sech}\left[(2n + 1)\frac{\pi K'}{2K}\right] \cos\left[\frac{(2n + 1)\pi\omega t}{2K}\right], \tag{7}$$

where $K(k)$ denotes the complete elliptic integral of the first kind, $K' = K(k')$, k' is the complementary elliptic modulus which satisfy the relation $k'^2 = 1 - k^2$. Further information on elliptic functions can be found in [13, 14]. Thus the Melnikov function takes the form

$$\begin{aligned} M(k, \omega; t_0) &= -4A\int_{-\infty}^{+\infty} cn(\omega(\tau + \tau_0)) \operatorname{sech}^2(\tau) \tanh\tau\,d\tau - 4\beta\int_{-\infty}^{+\infty} \operatorname{sech}^2(\tau)\,d\tau \\ &= -8\beta - 4A\frac{1}{k}\frac{\pi}{K}\sum_{n=0}^{n=\infty} \operatorname{sech}\left[(2n + 1)\frac{\pi K'}{2K}\right] \cos\left[\frac{(2n + 1)\pi\omega t_0}{2K}\right] \\ &\quad \times \int_{-\infty}^{+\infty} \operatorname{sech}^2\tau \tanh\tau \sin\left[(2n + 1)\frac{\pi\omega\tau}{2K}\right] d\tau. \end{aligned} \tag{8}$$

After evaluation of this last integral we may write the Melnikov function as

$$M(k, \omega; t_0) = -8\beta + 8AJ(k, \omega; t_0), \tag{9}$$

where $J(k, \omega; t_0)$ is defined as

$$\begin{aligned} J(k, \omega; t_0) &= \frac{1}{2k}\frac{\pi^4\omega^2}{8K^3}\sum_{n=0}^{\infty}(2n + 1)^2 \operatorname{sech}\left[(2n + 1)\frac{\pi K'}{2K}\right] \\ &\quad \times \operatorname{csch}\left[\frac{(2n + 1)\pi^2\omega}{4K}\right] \sin\left[\frac{(2n + 1)\pi\omega t_0}{2K}\right]. \end{aligned} \tag{10}$$

The condition for the homoclinic tangencies to occur is obtained when the Melnikov function changes sign at some t_0, and in our case this implies that

$$\frac{\beta}{A} = J(k, \omega), \tag{11}$$

where now $J(k, \omega)$ is

$$J(k, \omega) = \frac{1}{2k} \frac{\pi^4 \omega^2}{8K^3} \sum_{n=0}^{\infty} (2n + 1)^2 \operatorname{sech}\left[(2n + 1)\frac{\pi K'}{2K}\right] \operatorname{csch}\left[\frac{(2n + 1)\pi^2 \omega}{4K}\right]. \tag{12}$$

The above relation indicates the onset of homoclinic bifurcations, and for

$$\frac{\beta}{A} > J(k, \omega), \tag{13}$$

the crossing of the stable and the unstable manifolds of the saddle point $(\pi, 0)$ for the Poincaré map occurs giving rise to homoclinic chaos.

3. LIMITING CASES FOR THE THRESHOLD CONDITION

Equation (11) gives us the threshold function for the parameters A and β, responsible for the modulation and for the dissipation, respectively, to have a homoclinic bifurcation. In our case this function depends not only on the frequency of the modulation, as usually happens when the perturbation is harmonic, *but also on the elliptic modulus*, which is responsible for the shape of the periodic pulses used to modulate the pendulum. As k is bounded, i.e., $k \in [0, 1]$, it is natural to analyse equation (11) for the two limiting cases.

For the case in which $k \to 0$, the complete elliptic integral of the first kind has the limiting value $K(k) \to \pi/2$ and then the function $J_1(k, \omega)$ is given by

$$\lim_{k \to 0} J(k, \omega) = \frac{\pi \omega^2}{4} \operatorname{csch}\left[\frac{\pi \omega}{2}\right]. \tag{14}$$

Inserting this value into equation (11) provides us with the following result for the occurrence of homoclinic tangencies

$$\frac{\beta}{A} = \frac{\pi \omega^2}{4} \operatorname{csch}\left[\frac{\pi \omega}{2}\right], \tag{15}$$

which obviously coincides with the results in [11], where the pendulum is harmonically modulated. Taking into account that the Jacobi elliptic function *cn* converges to the cos circular function in this limit case, we recover the equation for the harmonically modulated pendulum as in [11]

$$\ddot{x} + \varepsilon \beta \dot{x} + (1 + \varepsilon A \cos \omega t) \sin x = 0. \tag{16}$$

The other limit we consider is $k \to 1$. Then, using equations (10) and (11), and remembering that the period for the Jacobi elliptic functions is $T = 4K/\omega$, we conclude that the condition for the homoclinic tangencies is given by

$$\frac{\beta}{A} = \frac{4\pi^2}{T^2} \sum_{n=0}^{\infty} (2n + 1) \operatorname{csch}\left[\frac{(2n + 1)\pi^2}{T}\right]. \tag{17}$$

This limiting case implies for the period that $T \to \infty$, and consequently that $\omega \to 0$. The result of all this is that we can never find parameters A and β satisfying the condition for the homoclinic tangencies and finally this makes impossible the appearance of chaos. In this second limiting case, the equation of motion reads

$$\ddot{x} + \varepsilon\beta\dot{x} + (1 + \varepsilon A)\sin x = 0, \tag{18}$$

which is the equation of the pendulum with dissipation, and clearly does not possess chaotic solutions.

In fact, the functional form that takes our conditions for homoclinic tangencies in equations (15) and (17) are very similar to the ones encountered for the Duffing oscillator excited externally [5]. This suggests that keeping constant the parameters and modifying only the elliptic modulus, i.e., the shape of the periodic perturbation, we may go from a chaotic state to a periodic state and vice versa.

A similar analysis as the one carried out in [5, 6] concerning the effect of a perturbation on an unstable limit cycle, which is modelled through the mapping $x_{n+1} = (\lambda + \varepsilon f_n)x_n$ $\lambda > 1$, can be done in our case, just taking into consideration that for the perturbation we have considered, $\langle cn \rangle = 0$ and $\langle cn^2 \rangle = 1 - (E(k) - K(k))/k^2K(k)$, where $E(k)$ is the complete elliptic integral of the second kind. This allows us to see how altering the elliptic modulus can change the sign of the Lyapunov exponent, modifying the stability of the orbit.

4. CONCLUSIONS

We have extended the study of the parametrically excited pendulum to the case when nonharmonic perturbations are present. We have found the general threshold condition for the occurrence of homoclinic bifurcations, with the help of the Melnikov method. It may be noted that this only roughly approximates to the experimental results, due to the perturbative nature of the method. This threshold value depends in our case, on the control parameters of the pendulum, the frequency of the excitation as well as on the elliptic modulus, which plays a fundamental importance in the analysis. We have considered the limiting values of this condition in relation to its dependence on the elliptic modulus. The analysis shows that through the control of the shape of the time-periodic signals used to perturb the system, we can move from a chaotic state to a periodic state, having fixed the rest of the parameters. Besides the intrinsic theoretical interest of this analysis, it seems also to be a good strategy for the experimentalist to check these new possible transitions order-chaos-order of the parametrically excited pendulum.

REFERENCES

1. L. Fronzoni, M. Giocondo and M. Pettini, Experimental evidence of suppression of chaos by resonant parametric perturbations, *Phys. Rev. A* **43**, 6483 (1991).
2. R. Lima and M. Pettini, Suppression of chaos by resonant parametric perturbations, *Phys. Rev. A* **41**, 726 (1990); *Phys. Rev. E* **47**, 4630 (1993).
3. G. Cicogna and L. Fronzoni, Modifying the onset of homoclinic chaos: application to a bistable potential, *Phys. Rev. E* **47**, 4585 (1993).
4. Y. Kivshar, F. Rödelspeger and H. Benner, Suppression of chaos by nonresonant parametric perturbations, *Phys. Rev. E* **49**, 319 (1994).
5. R. Chacón and J. Díaz Bejarano, Routes to suppressing chaos by weak periodic perturbations, *Phys. Rev. Lett.* **71**, 3103 (1993).
6. R. Chacón, Inhibition of chaos in hamiltonian systems by periodic pulses, *Phys. Rev. E* **50**, 750 (1994).
7. P. Bryant and K. Wiesenfeld, Suppression of period-doubling and nonlinear parametric effects in periodically perturbed systems, *Phys. Rev. A* **33**, 2525 (1986).
8. R. Saravanan, O. Narayan, K. Banerjee and J. K. Bhattacharjee, Chaos in a periodically forced Lorentz system, *Phys. Rev. A* **31**, 520 (1985).
9. J. B. McLaughlin, Period-doubling bifurcations and chaotic motion for a parametrically forced pendulum, *J. Stat. Phys.* **24**, 375 (1981).

10. J. Starrett and R. Tagg, Control of a chaotic parametrically driven pendulum, *Phys. Rev. Lett.* **74**, 1974 (1995).
11. B. P. Koch and R. W. Leven, Subharmonic and homoclinic bifurcations in a parametrically forced pendulum, *Physica D* **16**, 1 (1985).
12. S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*. Springer, Berlin (1990).
13. P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Physicists*. Springer, Berlin (1954).
14. D. Lawden, *Elliptic Functions and Applications*. Springer, Berlin (1989).