Pergamon

Chaos, Solitons & Fractals Vol. 9, No. 6, pp. 995–1003, 1998

Printed in Great Britain
0960-0779/98 $19.00 + .00

PII: S0960-0779(97)00181-1

Subharmonic Bifurcations in a Pendulum Parametrically Excited by a Non-harmonic Perturbation

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(Accepted 14 November 1997)

Abstract—A pendulum whose support is subjected to a periodic non-harmonic oscillation in the vertical direction is considered. The subharmonic Melnikov functions for the oscillating and for the rotating motions are explicitly constructed. It is shown that both functions converge towards the homoclinic Melnikov function and furthermore all the results for the harmonic perturbation are recovered.

1. INTRODUCTION

A well known characteristic precursor to chaotic motion is the appearance of subharmonics. In this paper we focus the attention on the subharmonic motions, either oscillations or rotations, of a pendulum whose support is subjected to a vertical non-harmonic periodic oscillation. A perturbation of this kind using elliptic functions has been considered by different authors [1–6] and, among other interesting aspects of its use has the advantage that introduces a new extra parameter and its Fourier spectrum contains more peaks than the ordinary trigonometric functions. Moreover, elliptic functions contain as limiting cases, the trigonometric functions. For this reason sometimes they have been called generalized sine or cosine. Besides that, they are the *natural solutions* of many nonlinear dynamical systems, the pendulum included and other polynomials nonlinear oscillators such as Duffing oscillator. In most of these references, [1–5], it has been stressed the fact that using this kind of perturbation transitions from periodic motion into chaotic motion and vice versa are possible. They have also been used to introduce the concept of geometrical resonance [5] as a way of explaining a chaos eliminating mechanism. However, in [6] an alternative way of switching among periodic orbits of different periodicity, once the rest of the parameters are fixed, by simply varying the elliptic parameter of the perturbation is introduced. In any case the use of this driving is not well understood and it possesses many aspects still not investigated.

The parametrically excited pendulum using a harmonic forcing has received considerable attention throughout the years. Among other authors, it has been studied analytically, numerically and experimentally by [7–10]. It may be said that it constitutes a paradigm for a nonlinear dynamical system parametrically perturbed and it is also connected to the nonlinear Mathieu equation. The boundaries of subharmonic and homoclinic bifurcations for the harmonically driven pendulum were explicitely calculated on the basis of Melnikov theory by [9]. They obtained explicit formulas for the subharmonic bifurcations corresponding to oscillating and rotating motions and they showed that the homoclinic bifurcation is the limit of a sequence of subharmonic saddle-node bifurcations, the oscillating converging from below and the rotational converging

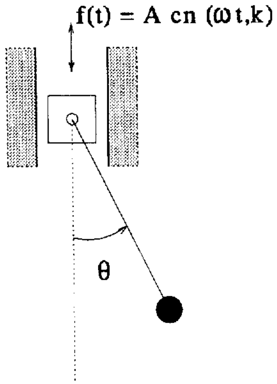

Fig. 1. Pendulum with a periodic non-harmonic oscillating vertical support.

from above. Our goal here is to evaluate the subharmonic Melnikov functions for the oscillating and rotating motions of the pendulum perturbed by an elliptic function and to study the existing relationship to the homoclinic bifurcations which were previously analyzed in [4]. Consequently this allows us to predict that the occurrence of subharmonic bifurcations as a parameter is varied, showing the relation between homoclinic and subharmonic bifurcations and the corresponding connection between periodic motion and complicated behaviour, which is clearly shown. Our results contain, as a particular case, the previous results by Koch and Leven [9].

2. PARAMETRICALLY DRIVEN PENDULUM

We consider a pendulum with a periodic oscillating support in the vertical direction, see Fig. 1. The periodic oscillation is provided by the Jacobi cosine amplitude elliptic function, $cn(\omega t,k)$, of frequency $\omega = 4K(k)/T$ and elliptic modulus k. The equations of motion for the parametrically forced pendulum with linear dissipation are given by

$$\ddot{x} + \epsilon\beta\dot{x} + (1 + \epsilon A cn(\omega t,k))\sin x = 0. \tag{1}$$

Written as a system of first order differential equations reads

$$\begin{aligned} \dot{x} &= v, \\ \dot{v} &= -\epsilon\beta v - (1 + \epsilon A cn(\omega t,k)\sin x, \end{aligned} \tag{2}$$

where $(x,v)\in S^1 \times R$, and β, A, ω are positive constants, with $0 < \epsilon \leq 1$.

The Jacobi elliptic functions are periodic functions of period $T = 4K(k)$, where $K(k)$ is the complete elliptic integral of the first kind which is defined as

$$K(k) = \int_0^{2\pi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}. \tag{3}$$

For a better understanding of the driving we use, Fig. 2 shows the dependence versus time of the $cn(t,k)$ for some values of k, while Fig. 3 depicts the Fast Fourier Transform, where an increasing number of peaks is shown for $k \neq 0$. The function $cn(t,k)$ has two limiting values, cos t as $k \to 0$ and sech$t$ when $k \to 1$. Further information on elliptic functions can be found in [11, 12].

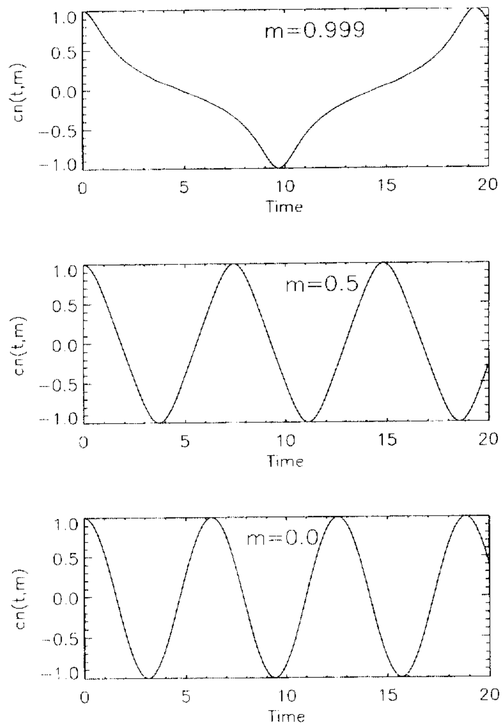

Fig. 2. The figure shows the variation of the cosine amplitude Jacobi elliptic function versus time for different values of $m = k^2$. Note that this is the relationship existing between the elliptic modulus k and the elliptic parameter m.

For completeness we review in this section the results for the homoclinic bifurcations, which appear in [4] and show the explicit solutions of the unperturbed pendulum for the different motions it possesses. The unperturbed pendulum has a Hamiltonian which may be written as

$$H = \frac{v^2}{2} - \cos x. \tag{4}$$

The phase space of the pendulum is 2π-periodic with hyperbolic saddles in $(\pm\pi,0)$ and an elliptic center in (0,0). There are three different kind of orbits for the unperturbed pendulum: rotations, oscillations and separatrix motion. The rotations are the unbounded motions, the oscillations are the bounded ones and the separatrix motion correspond to an oscillation of infinite period. Both hyperbolic saddles can be identified constructing a cylindrical phase space, and subsequently we speak about homoclinic connections and homoclinic motions, which correspond to the separatrix motion, instead of the heteroclinic ones. The solutions for the oscillating orbits can be expressed in terms of the Jacobi elliptic functions as

$$(x_{osc}(t),v_{osc}(t)) = (2\sin^{-1}(ksn(t,k)),2kcn(t,k)). \tag{5}$$

The elliptic modulus k is related to the energy or Hamiltonian of the unperturbed pendulum through the expression

$$H = 2k^2 - 1, \tag{6}$$

in such a way that it labels the different orbits. From the expression above, we can obtain the solutions for the homoclinic orbit, taking the limit $k \to 1$ for the elliptic modulus. A physical and

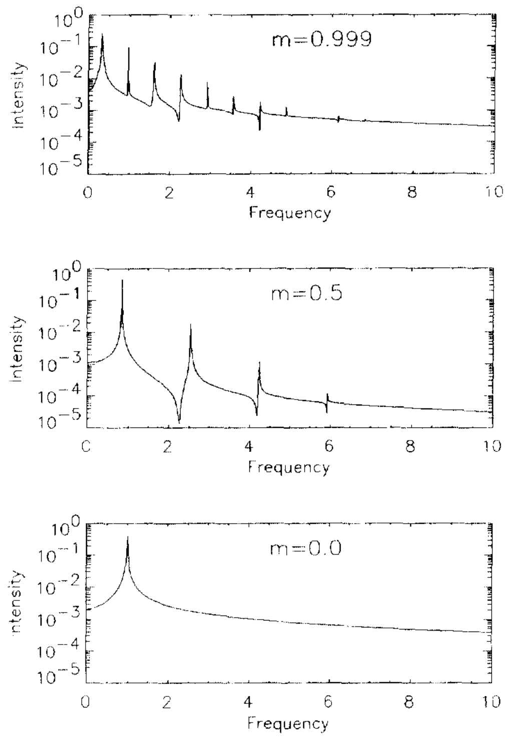

Fig. 3. Fast Fourier Transform of the cosine amplitude Jacobi elliptic function for different values of the elliptic parameter $m = k^2$.

intuitive picture of the meaning of k is obtained by thinking in the unperturbed orbits in phase space inside a separatrix orbit. The parameter k is used to label energy of the orbits inside the separatrix. For orbits with energy very small in absolute terms, $k \to 0$, the complete elliptic integral of the first kind $K(k) \to \pi/2$ and consequently the period becomes $T \to 2\pi$. This corresponds to the period for the linear oscillations around the elliptic fixed point. However for values of the energy close to the separatrix, which means $k \to 1$, the complete elliptic integral of the first kind diverges logarithmically and the period becomes infinity. The homoclinic solutions are then

$$(x_{sx}(t), v_{sx}(t)) = (2 \tanh t \operatorname{sech} t, 2 \operatorname{sech} t). \tag{7}$$

The Melnikov function has to be evaluated for the homoclinic orbit and takes the following expression in our case

$$M(k,\omega;t_0) = \int_{-\infty}^{+\infty} v_{sx}(t-t_0)\{-Acn(\omega t,k)\sin(x(t-t_0)) - \beta v_{sx}(t-t_0)\}\, dt. \tag{8}$$

To calculate this integral, which involves the cosine Jacobi elliptic function, we consider its Fourier expansion, which is given by

$$cn(\omega t, m) = \frac{1}{k}\frac{\pi}{K}\sum_{j=0}^{\infty} \sec h\left[(2j+1)\frac{\pi K'}{2K}\right]\cos\left[\frac{(2j+1)\pi\omega t}{2K}\right], \tag{9}$$

where $K(k)$ denotes the complete elliptic integral of the first kind, $K' = K(k')$ and k' is the complementary elliptic modulus which satisfy the relation $k'^2 = 1 - k^2$. Thus the homoclinic Melnikov function takes the form

$$M(k,\omega;t_0) = -4A\int_{-\infty}^{+\infty} cn(\omega(\tau+\tau_0))\sec h^2(\tau)\tanh\tau d\tau - 4\beta\int_{-\infty}^{+\infty}\sec h^2(\tau)d\tau$$

$$= -8\beta - 4A\frac{1}{k}\frac{\pi}{K}\sum_{j=0}^{\infty}\sec h\left[(2j+1)\frac{\pi K'}{2K}\right]\cos\left[\frac{(2j+1)\pi\omega t_0}{2K}\right]$$

$$\times\int_{-\infty}^{+\infty}\sec h^2\tau\tanh\tau\sin\left[(2j+1)\frac{\pi\omega\tau}{2K}\right]d\tau. \tag{10}$$

After evaluation of this last integral we may write the Melnikov function as

$$M(k,\omega;t_0) = -8\beta + 8AJ_1(k,\omega;t_0), \tag{11}$$

where $J_1(k,\omega;t_0)$ is defined as

$$J_1(k,\omega;t_0) = \frac{1}{2k}\frac{\pi^4\omega^2}{8K^3}\sum_{j=0}^{\infty}(2j+1)^2\sec h\left[(2j+1)\frac{\pi K'}{2k}\right]csc\,h\left[\frac{(2j+1)\pi^2\omega}{4K}\right]$$

$$\times\sin\left[\frac{(2j+1)\pi\omega t_0}{2K}\right]. \tag{12}$$

The condition for the homoclinic tangencies to occur is produced when the Melnikov function changes sign at some t_0, and in our case this implies that

$$\frac{\beta}{A} \geqq J_1(\mathrm{k},\omega), \tag{13}$$

where $J_1(k,\omega)$ is defined as

$$J_1(k,\omega) = \frac{1}{2k}\frac{\pi^4\omega^2}{8K^3}\sum_{j=0}^{\infty}(2\mathrm{j}+1)^2\sec h\left[(2j+1)\frac{\pi K'}{2K}\right]csc\,h\left[\frac{(2j+1)\pi^2\omega}{4K}\right]. \tag{14}$$

The equality indicates the onset of homoclinic bifurcations above this value, the crossing of the stable and the unstable manifolds of the saddle point $(\pi,0)$ for the Poincaré map gives rise to homoclinic chaos.

3. SUBHARMONIC MELNIKOV FUNCTIONS

The subharmonic Melnikov theory and its bifurcations has deserved attention by several authors [9, 13–15]. Our main interest here is to evaluate the subharmonic Melnikov functions for the oscillating and for the rotation motions in the pendulum parametrically driven by the cosine amplitude Jacobi elliptic function and to relate them to the homoclinic ones.

The subharmonic Melnikov function is defined as

$$M^{m/n}(t_0) = \int_0^{mT} v(t-t_0,k)\,[-A\,cn(\omega\tau,k)\sin(x(t-t_0),k) - \beta v(t-t_0,k)]. \tag{15}$$

We look for periodic orbits in the perturbed system that are in resonance with the external forcing and that satisfy the resonance relation $n\cdot T^\alpha = m\cdot T$, where T^α denotes the period of the resonant orbits. The period of the external perturbation is $T=[4K(k)/\omega]$. The resonance condition for the oscillating motions is $4K(k) = T\cdot m/n$, from where the k values follow, and m and n are relatively prime numbers. For the rotating motions the resonance condition is

$$2\frac{K\left(\frac{1}{k}\right)}{k} = T\cdot\frac{m}{n}.$$

3.1. *Oscillating motions*

The calculations for the oscillating motion are given by

$$M_{osc}^{m/n}(t_0) = -4Ak^2\int_0^{mT} cn(\omega\tau,k)dn(t-t_0,k)sn(t-t_0,k)cn(t-t_0,k)dt -$$

$$-4\beta k^2\int_0^{mT} cn^2(t-t_0,k)dt, \quad (16)$$

where $sn(\cdot)$, $cn(\cdot)$ and $dn(\cdot)$ are Jacobi elliptic functions [12]. We take into account again the Fourier expansion of the cosine amplitude Jacobi elliptic function, (equation (9)), and we then have

$$M_{osc}^{m/n}(t_0) = -4Ak^2\frac{\pi}{kK}\sum_{j=0}^{\infty}\sec h\left[(2j+1)\frac{\pi K'}{2K}\right]$$

$$\times\int_0^{mT}\cos\left[\frac{(2j+1\pi\omega t_0}{2K}\right]dn(t-t_0,k)sn(t-t_0,k)cn(t-t_0,k)dt$$

$$-4\beta k^2\int_0^{mT} cn^2(t-t_0,k)dt. \quad (17)$$

The first integral vanishes except for $n = 1$ and even m. Thus we arrive to

$$M_{osc}^{m}(t_0) = AJ_2(k,\omega;t_0) = 16\beta\{E(k)-k'^2K(k)\}, \quad (18)$$

where $E(k)$ is the complete elliptic integral of the second kind and $J_2(k,\omega;t_0)$ is defined as

$$J_2(k,\omega;t_0) = \frac{\pi^4\omega^2}{kK^3}\sum_{n=0}^{\infty}(2j+1)^2\sec h\left[(2j+1)\frac{\pi K'}{2K}\right]csc\, h\left[\frac{(2j+1)\pi\omega K'}{2K}\right]$$

$$\times\sin\left[\frac{(2j+1)\pi\omega t_0}{2K}\right]. \quad (19)$$

Thus the condition for the ocurrence of bifurcations for oscillating motions is given by

$$\frac{A}{\beta} = \frac{16\{E(k)-k'^2K(k)\}}{J_2(k,\omega)} = R_{osc}^{m}, \quad (20)$$

where $J_2(k,\omega)$ is defined as

$$J_2(k,\omega) = \frac{\pi^4\omega^2}{kK^3}\sum_{n=0}^{\infty}(2j+1)^2\sec h\left[(2j+1)\frac{\pi K'}{2K}\right]csc\, h\left[\frac{(2j+1)\pi\omega K'}{2K}\right] \quad (21)$$

and where m take even values.

3.2. *Rotating motions*

The solutions for the rotating orbits can be expressed in terms of the Jacobi elliptic functions as

$$(x_{rot}(t), v_{rot}(t)) = (2\sin^{-1}(sn(kt, 1/k)), 2k\,dn(kt, 1/k)). \tag{22}$$

For the rotating part we have the following expression:

$$M^{m/n}_{rot}(t_0) = -4Ak\int_0^{mT} cn\left(\omega\tau, k) dn(k(t-t_0), \frac{1}{k}\right) sn\left(k(t-t_0), \frac{1}{k}\right) cn\left(k(t-t_0), \frac{1}{k}\right) dt$$

$$-4\beta k^2 \int_0^{mT} dn^2\left(k(t-t_0), \frac{1}{k}\right) dt.$$

Using again the Fourier expansion for the cosine amplitude (equation (9)), we then have

$$M^{m/n}_{rot}(t_0) = -4Ak\frac{\pi}{kK}\sum_{j=0}^{\infty} \sec h\left[(2j+1)\frac{\pi K'}{2K}\right]$$

$$\times \int_0^{mT} \cos\left[\frac{(2j+1)\pi\omega t}{2K}\right] dn\left(k(t-t_0), \frac{1}{k}\right) sn\left(k(t-t_0), \frac{1}{k}\right) cn\left(k(t-t_0), \frac{1}{k}\right)$$

$$-4\beta k^2 \int_0^{mT} \mathrm{dn}^2\left(k(t-t_0), \frac{1}{k}\right) dt. \tag{24}$$

The first integral vanishes except for the case $n = 1$ and the result of the calculation is

$$M^{m}_{rot}(t_0) = AJ_3(k, \omega; t_0) - 8\beta k E\left(\frac{1}{k}\right), \tag{25}$$

where $J_3(k, \omega; t_0)$ is defined as

$$J_3(k, \omega; t_0) = \frac{\pi^4\omega^2}{2K^3}\sum_{n=0}^{\infty} (2j+1)^2 \sec h\left[(2j+1)\frac{\pi K'}{2K}\right] csc\,h\left[\frac{(2j+1)\pi\omega K'\left(\frac{1}{k}\right)}{2kK}\right]$$

$$\times \sin\left[\frac{(2j+1)\pi\omega t_0}{2K}\right]. \tag{26}$$

From this equation the condition for the ocurrence of the subharmonic bifurcations is then given by

$$\frac{A}{\beta} = \frac{8kE\left(\frac{1}{k}\right)}{J_3(k, \omega)} = R^{m}_{rot}, \tag{27}$$

where $J_3(k, \omega)$ is defined as

$$J_3(k,\omega) = \frac{\pi^4\omega^2}{2K^3}\sum_{n=0}^{\infty}(2j+1)^2 \sec h\left[(2j+1)\frac{\pi K'}{2K}\right] csc\, h\left[\frac{(2j+1)\pi\omega K'\left(\frac{1}{k}\right)}{2kK}\right]. \tag{28}$$

3.3. *Relationship between oscillating and rotating subharmonic and homoclinic melnikov functions*

We are going now to study the limiting behaviour of the subharmonic functions $M^m_{osc}(t_0)$ and $M^m_{rot}(t_0)$ for the case in which $m \to \infty$, or equivalently $k \to 1$.

According to the asymptotic values of the complete elliptic integrals and taking into account that as it can be seen $\lim_{m\to\infty} J_2(k,\omega) = 8J_1(k,\omega)$, then this implies that

$$\lim_{m\to\infty} M^m_{osc}(t_0) = 8AJ_1(k,\omega) = 16\beta = 2M(t_0) \tag{29}$$

On the other hand for the rotating motions we have that $\lim_{m\to\infty} J_3(k,\omega) = 4J_1(k,\omega)$ what makes that

$$\lim_{m\to\infty} M^m_{rot}(t_0) = 4AJ_1(k,\omega) = 8\beta = M(t_0). \tag{30}$$

Finally we conclude that

$$\frac{1}{2}\lim_{m\to\infty} M^m_{osc}(t_0) = \lim_{m\to\infty} M^m_{rot}(t_0) = M(t_0). \tag{31}$$

It clearly follows from this expression that the homoclinic bifurcation is the limit of a sequence of subharmonic saddle-node bifurcations. The oscillating subharmonics converging from below and the rotating subharmonics converging from above.

3.4. *Limiting case*

Once we have obtained the results for the pendulum perturbed by an elliptic function, we want to address the limiting case for which $k \to 0$. The values the elliptic parameter may take are comprised between $0 \le k \le 1$. For small values of the elliptic parameter the cosine amplitude Jacobi elliptic function becomes the cosine trigonometric function. The limit $k \to 0$ of our model equation (equation (1)) coincides with the model used by Koch and Leven [9] for the pendulum parametrically excited by a trigonometric cosine function, then our results for small values of k need to coincide with those obtained by Koch and Leven [9]. For small values of k, the complete elliptic integral of first kind has the limiting value $K(k) \to \pi/2$, and after some algebra it may be proved for the boundary subharmonic functions $J_2(k,\omega)$ and $J_3(k,\omega)$ to have the following limiting values

$$\lim_{k\to 0} J_2(k,\omega) = \frac{4\pi\omega^2}{\sin h(\omega K')}, \tag{32}$$

$$\lim_{k\to 0} J_3(k,\omega) = \frac{2\pi\omega^3}{\sin h\left(\frac{wK'\left(\frac{1}{k}\right)}{k}\right)}. \tag{33}$$

Consequently introducing the limiting values of the functions $J_2(k,\omega)$ and $J_3(k,\omega)$, equations (32) and (33), into the results which give the parameter values for the occurrence of subharmonic bifurcations, equations (20) and (27), all the results given in [9] are recovered.

4. CONCLUSIONS

Subharmonic Melnikov functions for the pendulum parametrically excited by periodic non-harmonic pulses are explicitly constructed. They tell us for which system parameter values the occurence of subharmonic bifurcations takes place. It is shown that the corresponding functions for oscillating and for rotating motions converge towards the homoclinic function, the oscillating from below and the rotating from above. Moreover when the elliptic modulus is very small the perturbation we use, coincides with the harmonic perturbation used by Koch and Leven, and consequently all their results are recovered. Thus, the work done on homoclinic bifurcations by [4] on the same model is also extended, and the results we have obtained represent a generalization of previous results which are contained in ours as a particular case.

Acknowledgement—This work was supported by the Ministry of Education and Culture, Spain, project DGES PB-0123.

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