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Energy dissipation in a nonlinearly damped Duffing oscillator

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Abstract

In this paper, we study the effect of including a nonlinear damping term proportional to the power of the velocity in the dynamics of a double-well Duffing oscillator. In particular, we focus our attention in understanding how the energy dissipates over a cycle and along the time, by the use of different tools of analysis. Analytical and numerical results for different damping terms are shown, and the presence of a discontinuity and an inversion of behavior depending on the initial energy are discussed. An averaged power loss in a period is defined, showing similar characteristics as the energy dissipation over a cycle, although no discontinuity is present. The discontinuity gap which appears for the energy dissipation at a certain value of the initial energy decreases as the power of the damping term increases and an associated scaling law is found. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The phenomenon of dissipation is ubiquitous in physical and mechanical systems, always meaning a loss of energy due to friction forces. Friction has been a topic of technological attention since the early days of our civilization, and its interest in scientific and engineering research remains today [1]. Most systems are subject to a different combination of terms of damping, each one of them with different effects. These dissipative forces may act on a system in the form of dry friction, such as Coulomb friction, which is typically a nonlinear damping mechanism, and which has deserved much attention recently in relation to its physical origins [2,3]. Other typical sources of friction is drag or aerodynamics friction, which is typically proportional to the square of the velocity, and viscous forces which are linear in the velocity. The main advantage of the viscous linear damping term comes from its simplicity, since the linearity of the governing differential equation, in the simple case of a harmonic oscillator, is not altered.

In ordinary applications of nonlinear dynamical systems such as the motion of an object through a fluid or gas, frictional and drag forces are usually very complex, and power-law dissipative terms in the velocity are used. However, one of the problems of considering these and other forms of damping, besides the common linear ones, is that they give rise to nonlinear terms in the governing differential equations even for the simplest system, without the

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possibility of obtaining exact solutions. The mathematical form of the friction forces is usually experimentally determined with the use of wind tunnels or water tanks. Among the possible models, one of the simplest empirical mathematical model of the dissipative force is taken as $f(v) \propto -v|v|^{p-1}$, where v represents the velocity, with its positive or negative sign, and $p \geq 0$. Notice that the absolute value of the velocity is needed to safeguard the fact that the dissipative force must be contrary to the motion of the system. Taking this fact into account, one can recover from this expression the important cases of Coulomb damping ($p = 0$), linear viscous damping ($p = 1$) and draglike friction ($p = 2$). A historical review of the applications of the Coulomb damping ($p = 0$) can be found in [1], whereas the extensively used case $p = 1$ and its applications are considered, for example in [4,5]. Phenomenological models accounting for nonlinear dissipation forces with higher values of p have been considered in some applied sciences, such as ship dynamics, vibration engineering, and motion of projectiles, among others [6–11]. A physical interpretation of these higher order damping terms may be given by assuming a power series expansion on the velocity of more complicated damping forces.

The effect of some physical parameters, such as wave amplitude and nonlinear damping, on the transient motions and the global system behavior of a capsizing ship was investigated in [7]. In particular, they use a model of a double-well and a single-well Duffing oscillator with a linear plus a quadratic damping term. The stability of a nonlinearly damped hard Duffing oscillator was analyzed in [11], and the role of nonlinear dissipation in soft Duffing oscillators in [12]. Moreover, in [13] it was shown that nonlinear damping could be used as a mechanism to suppress chaos. The effect of nonlinear dissipation on some properties of the dynamics of the universal escape oscillator, such as the threshold of period-doubling bifurcations, fractal basins boundaries and the destruction of the basins of attraction was considered in [14], and some analytical estimates of the nonlinear damping effects for the Duffing oscillator and for the pendulum, using Melnikov techniques, in [15]. A rather interesting example of application of this nonlinear damping appears in [16] in the context of dissipation in barred galaxies and the analysis of the mechanisms by which mass concentrations are produced, where a model of a quadratically damped pendulum is used. Dynamical friction of this sort is common in galactic dynamics [17].

A discussion of a general analysis of damping appears in [6] in the context of the naval architecture literature, including combinations of linear plus quadratic damping terms, linear plus cubic damping terms or even combinations of linear plus quadratic plus cubic terms. A general damping function of the form

$$D(\dot{x}) = \sum_{p=1}^{N} b_p \dot{x}|\dot{x}|^{p-1} \tag{1}$$

is used which includes a general combination of different damping terms ($b_p \geq 0$) that are powers in the velocity. In our analysis on the energy dissipation, we consider terms of the form $b\dot{x}|\dot{x}|^{p-1}$, and our results apply when a combination of terms of the above form is used.

From the nonlinear dynamics point of view, a dissipative system with a linear dissipative term has a rate of contraction of phase space which is constant. This is not the case when the dissipative force is nonlinear and is proportional to a power of the velocity, since it implies that the rate of contraction of phase space is velocity dependent. The idea of the volume contraction in phase space is somehow related to the logarithmic decrement of linear oscillators with linear damping and it might be also a generalization of the same idea for general nonlinear oscillators. It is common practice in physics and engineering to analyze the effects of damping by measuring the decay of free oscillations and the loss of energy in physical systems. A discussion of damping in nonlinear oscillators in the context of nonlinear dynamics appears in [5].

In the present work, we analyze the effect of using nonlinear damping terms on the loss of energy over a cycle and as a function of time, and the decay of the free oscillations for the nonlinear double-well Duffing oscillator. Commonly, this nonlinear oscillator has been analyzed with the inclusion of a simple viscous (linear) damping term, however, we have here a model of a nonlinear oscillator with a *nonlinear* damping term. This system plays a role of a paradigm in nonlinear dynamics and for this reason we aim at carrying out analytical and numerical computations which provide information on how the energy dissipates for a better understanding of its physical mechanisms.

The organization of the paper is as follows. A general overview of how the energy dissipates, including the analysis of the decay of the free oscillations, and a discussion on the results (analytical and numerical) concerning the loss of the energy over a cycle, are described in some detail in Section 2. In Section 3, we study how the energy dissipates as a function of time and analyze the logarithmic decrement associated to the decay of the amplitude of the oscillations. Since the energy dissipated over a cycle depends strongly on the period of the motion, we define the *averaged power loss* and devote Section 4 to its study and to analyze some scaling laws which appear in the analysis of the energy dissipated over a cycle for different values of the damping exponents. In Section 5 we give the conclusions. And finally, in Appendixes A and B, the analytical solutions for the undamped and unforced Duffing oscillator and a list of integrals involving Jacobian elliptic functions used throughout this work, are included.

2. Overview of the energy dissipation

2.1. Description of the model and decay of the oscillations

As we mentioned earlier, our goal is to analyze the effect of nonlinear damping terms on the behavior of a nonlinear oscillator. In particular, we focus our attention on the nonlinear Duffing oscillator. We consider a general nonlinear dissipative force proportional to the power of the velocity of the form

$$F_p = -b\dot{x}|\dot{x}|^{p-1}, \tag{2}$$

where b ($b \ll 1$) is the damping coefficient, p (p real) the damping exponent and $\dot{x}$ stands for the velocity. Our purpose here is to estimate the energy dissipation of the Duffing oscillator when a dissipative term of the previous form is included. Thus, the general model equation for the nonlinearly damped Duffing oscillator reads

$$\ddot{x} + b\dot{x}|\dot{x}|^{p-1} - x + x^3 = F\cos\omega t, \tag{3}$$

where F is the amplitude and ω the frequency of the external perturbation. For the objectives of this paper, our interest is focused in the decay of the free oscillations and how the energy dissipates over a cycle and as a function of time, so we are interested in the unforced oscillator, i.e., we fix $F = 0$ throughout the whole paper.

One of the first consequences of the damping on a free oscillator is the decay of the amplitude of the oscillations. In order to observe and analyze this phenomenon in our case, we have numerically computed the solutions of the free nonlinearly damped Duffing oscillator for different values of the initial energy. We have numerically integrated Eq. (3) using a fourth-order Runge–Kutta integrator. The temporal evolutions of the amplitude $x(t)$ are plotted in Fig. 1, when $b = 0.01$, for different values of the damping exponent p and the energy E in order to visualize how the energy dissipates, and how it affects the amplitude of the oscillation. This figure shows six panels in three rows. The first row refers to the case of an orbit situated in the right well, and Fig. 1(a) shows the case of dry friction, where the energy dissipates rather quickly with an apparent linear decrement settling down at the bottom of the well. We can see in Fig. 1(b) for the same energy but with a cubic friction term that the energy takes longer time to decay. Note that one thing is how the energy dissipates as a function of time, and a different one is the dissipation over a cycle. Fig. 1(c) and (d) shows, for an orbit situated outside the well, that the amplitude decays faster for the case of dry friction than for the case of cubic friction. The last row, Fig. 1(e) and (f), corresponds to the case of an orbit of higher initial energy. For $p = 0$, the energy takes longer to decay than for $p = 3$, as in the previous cases. However, in this case, for $p = 0$ the energy dissipated over a cycle for $p = 0$ is lower than for $p = 3$, as we will see in the next section.

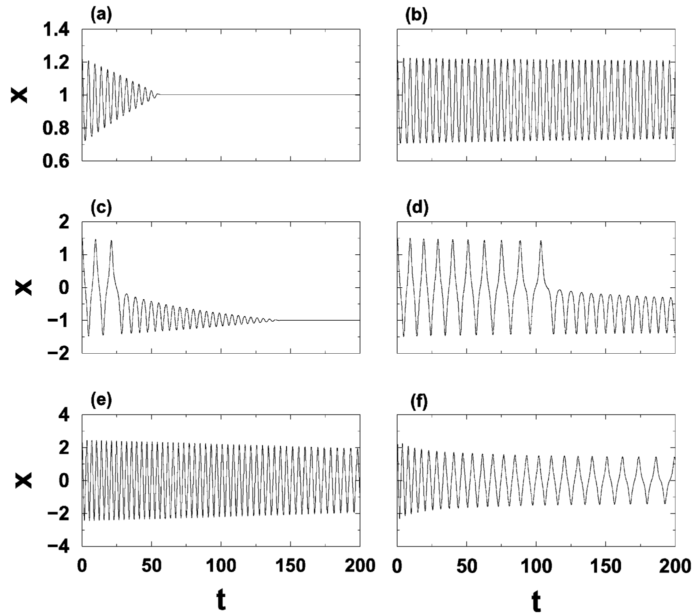

Fig. 1. Decay of the amplitude of the free oscillations of the nonlinearly damped Duffing oscillator $\ddot{x} + b\dot{x}|\dot{x}|^{p-1} - x + x^3 = 0$ for different damping exponents p and different values of the initial energy ($\alpha = \sqrt{1+4E}$): (a) $p = 0$ and $\alpha = 0.5$, (b) $p = 3$ and $\alpha = 0.5$, (c) $p = 0$ and $\alpha = 1.25$, (d) $p = 3$ and $\alpha = 1.25$, (e) $p = 0$ and $\alpha = 5$, (f) $p = 3$ and $\alpha = 5$.

2.2. Energy dissipation over a cycle

Now we are interested in evaluating how the energy dissipates over a cycle. Following the argumentation given in [4], the energy dissipated ΔE over one cycle of period T due to the damping force F_p is

$$\Delta E = \int_0^T F_p \dot{x}\, \mathrm{d}t. \tag{4}$$

For simplicity during all our calculations, we define the quantity

$$\Delta_p \equiv \frac{\Delta E}{-b} = \int_0^T \dot{x}^2 |\dot{x}|^{p-1}\, \mathrm{d}t, \tag{5}$$

where this last integral can be estimated analytically in some cases (when p is an integer), whereas otherwise numerical techniques are used. For integer values of the damping exponent p, all we need to evaluate this last integral is the amplitude $x(t)$ for the unforced and undamped oscillator and its period T. These expressions are given in Appendix A. All the necessary integrals involving Jacobian elliptic functions needed to compute these expressions have been included in Appendix B. We have carried out analytically the computations for the energy dissipated over one cycle Δ_p for the damping exponents $p = 0, 1, 2, 3$, which correspond to the cases of Coulomb damping, linear viscous damping, quadratic and cubic damping, respectively. All the results are displayed in Table 1. In all these expressions, the parameter α is defined in terms of the energy E of the orbit as $\alpha = \sqrt{1+4E}$, and $K(m)$ and $E(m)$ are the complete elliptic integrals of the first and second kind, respectively. Information concerning Jacobian elliptic functions is found in [18,19]. The values of Δ_p are computed for different orbits with initial energies below the local maximum and above the local maximum, which are referred in the table as regions I and II, respectively.

In Fig. 2, we have plotted the analytical results for Δ_p for integer values of p increasing from $p = 0$ to 3, (solid line), and we have also included the corresponding numerical results (symbols). As is observed, the agreement between the exact solutions and the numerical computations is excellent.

Table 1
This table shows the analytical results of the energy dissipation over a cycle Δ_p for different integer values of the damping exponent p for the nonlinearly damped Duffing oscillator $\ddot{x} + b\dot{x}|\dot{x}|^{p-1} - x + x^3 = 0$[a]

p	Region	Δ_p
0	I	$\Delta_0 = 2(\sqrt{1+\alpha} - \sqrt{1-\alpha})$
	II	$\Delta_0 = 4\sqrt{1+\alpha}$
1	I	$\Delta_1 = \frac{4}{3}\sqrt{\frac{1+\alpha}{2}}\left[(\alpha-1)K\left(\sqrt{\frac{2\alpha}{1+\alpha}}\right) + E\left(\sqrt{\frac{2\alpha}{1+\alpha}}\right)\right]$
	II	$\Delta_1 = \frac{4}{3}\sqrt{\alpha}\left[(\alpha-1)K\left(\sqrt{\frac{1+\alpha}{2\alpha}}\right) + 2E\left(\sqrt{\frac{1+\alpha}{2\alpha}}\right)\right]$
2	I	$\Delta_2 = \frac{4}{15}[(1+\alpha)^{3/2}(3\alpha-2) + (1-\alpha)^{3/2}(3\alpha+2)]$
	II	$\Delta_2 = \frac{8}{15}(1+\alpha)^{3/2}(3\alpha-2)$
3	I	$\Delta_3 = \frac{4}{35}\sqrt{\frac{1+\alpha}{2}}\left[(5\alpha^2-4)(\alpha-1)K\left(\sqrt{\frac{2\alpha}{1+\alpha}}\right) + 4(2\alpha^2-1)E\left(\sqrt{\frac{2\alpha}{1+\alpha}}\right)\right]$
	II	$\Delta_3 = \frac{4}{35}\sqrt{\frac{1+\alpha}{2}}\left[(5\alpha^2-3\alpha-4)(\alpha-1)K\left(\sqrt{\frac{1+\alpha}{2\alpha}}\right) + 8(2\alpha^2-1)E\left(\sqrt{\frac{1+\alpha}{2\alpha}}\right)\right]$

[a] The region I indicates the region inside the double well where the initial energy of an orbit is $-\frac{1}{4} < E < 0$, while the region II refers to an orbit with initial energy $E > 0$, i.e., outside the double well. The parameter α is related to the initial energy E of an orbit through the relationship $\alpha = \sqrt{1+4E}$. $K(m)$ and $E(m)$ are the complete elliptic integrals of first and second kind, respectively, where $0 < m < 1$ is the elliptic modulus. A plot of the energy dissipation over a cycle versus the initial energy for the different values of p considered here is represented in Fig. 2.

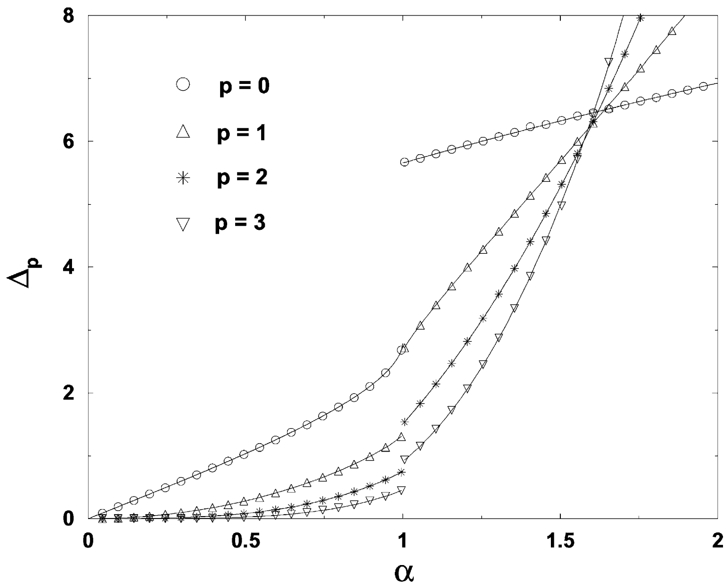

Fig. 2. Dependence of the loss of the energy over a cycle Δ_p versus the initial energy ($\alpha = \sqrt{1+4E}$), for several integer values of the damping exponent p. The symbols refer to the numerical computations, whereas solid lines represent the analytical calculations. The discontinuity at $\alpha = 1$ is a manifestation of the transition from the double well to the single well above the local maximum. Notice the inversion of the behavior around the value $\alpha = \sqrt{3}$.

After the observation of this figure, we can conclude that the energy dissipated over one cycle, Δ_p, increases as α increases for a fixed p. This can be explained if we realize that, given the dependence of the damping term with the velocity of the system (see Eq. (2)), it is obvious that the faster the system moves, the stronger the damping effect and therefore the energy dissipation is. This indicates that the energy dissipation over a period for a fixed p should increase with α, i.e., with the total energy, since the velocity of the system also increases when the total energy increases.

On the other hand, it is worth noticing how Δ_p behaves for a given α, when the damping exponent p varies. We observe that Δ_p decreases when p increases for a certain range of values of α, but it inverts its behavior when α is above a value around 1.7. The explanation of this fact can be analyzed again in terms of the damping term. We may consider two values of p, say p_1 and p_2, such that $p_1 > p_2$. It can be shown that the velocity of the orbit $\dot{x}$ is such that $|\dot{x}| < 1$ when $\alpha < \sqrt{3}$ and that only when $\alpha \geq \sqrt{3}$ the velocity is greater than 1 for some instants of time. Taking into account that the nonlinear dissipative forcing term F_p has a dependence on p given by Eq. (2), one would expect the damping effect to be weaker for p_2 than for p_1 when α is below a certain value close to $\sqrt{3}$ ($|\dot{x}| < 1$), whereas the opposite occurs when α is above this value ($|\dot{x}| > 1$). This implies that, roughly, Δ_p is a decreasing function of p for $\alpha < \sqrt{3}$ and an increasing function of p when $\alpha \geq \sqrt{3}$, which clearly corresponds to our results. Finally, the discontinuity at $\alpha = 1$ is understood if we realize that for values of $\alpha < 1$, the system is forced to move inside one of the two wells, whereas for values of $\alpha > 1$ the system has enough energy to move outside the wells (see Fig. 3). So, it is straightforward to see that a discontinuity in both the period (see Fig. 4), and the velocity of the system, and therefore, in Δ_p, must appear. It is interesting to note as well the gap of the curves at the discontinuity ($\alpha = 1$), which is smaller as the damping exponent p increases. A scaling law associated to this phenomenon will be discussed in Section 4.

For completeness, we have carried out numerical computations involving noninteger values of p. Some curves have been depicted in Fig. 5(a) and (b). In these figures, we have displayed Δ_p versus α for p ranging from 0 to 1 at intervals of 0.25 (Fig. 5(a)), and from $p = 1$ to 2 with the same interval (Fig. 5(b)). The dependence of Δ_p on α for noninteger values of p is very similar to that exhibited by the system in Fig. 2. This means that the energy loss increases when α increases for a given p, whereas Δ_p decreases when p

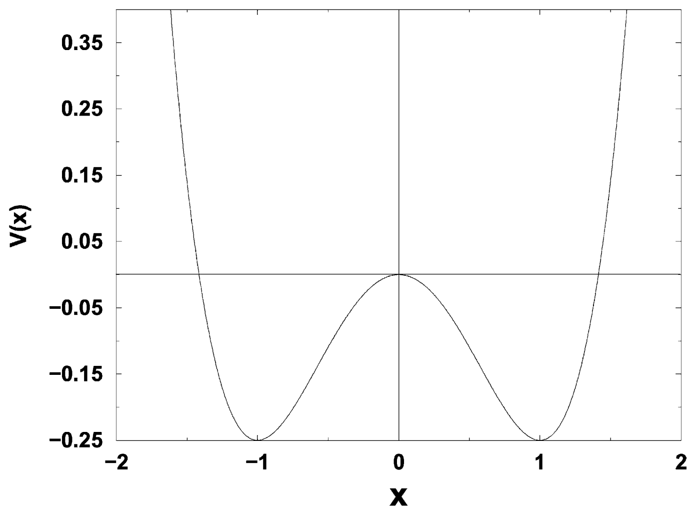

Fig. 3. Plot of the double well potential $V(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2$, associated to the Duffing oscillator. Motions inside the double well corresponds to initial energies $-\frac{1}{4} < E < 0$ and consequently $0 < \alpha < 1$. For values of the initial energy $E > 0$, $\alpha > 1$, the motions are outside the double well.

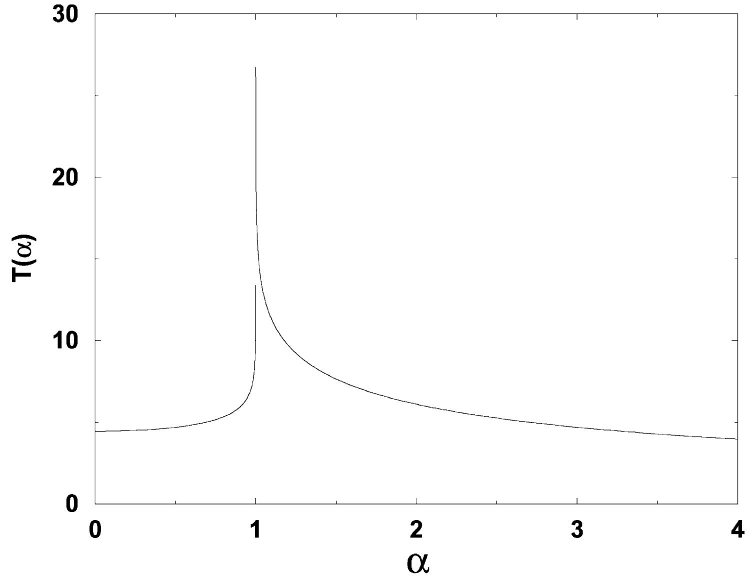

Fig. 4. Period of the free Duffing oscillator as a function of $\alpha = \sqrt{1+4E}$. Note that $T(\alpha) \to \infty$ both when $\alpha \to 1^+$ and $\alpha \to 1^-$, but simultaneously when $\alpha \to 1^+$ the value of $T(\alpha)$ is twice the value of $T(\alpha)$ when $\alpha \to 1^-$. This divergence is explained by the existence of a homoclinic orbit due to the presence of a hyperbolic fixed point in the phase space for $\alpha = 1$.

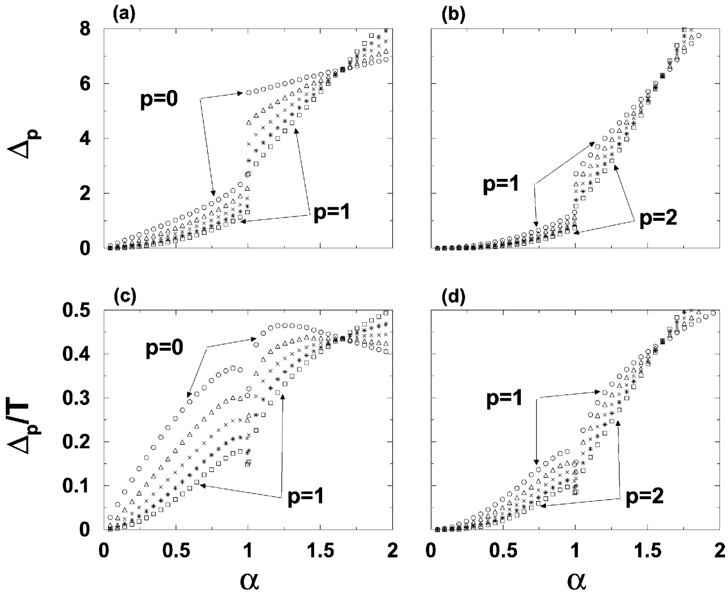

Fig. 5. (a) and (b) Energy dissipation losses over a cycle for the nonlinearly damped Duffing oscillator as a function of $\alpha = \sqrt{1+4E}$ for different real values of the damping exponent p. (c) and (d) Averaged power losses over a cycle for the nonlinearly damped Duffing oscillator as a function of α for different real values of the damping exponent p. In the left panels (a) and (c), circles stand for $p = 0$, triangles for $p = 0.25$, crosses for $p = 0.5$, stars for $p = 0.75$ and squares for $p = 1$. On the other hand, in the right panels (b) and (d), circles correspond to $p = 1$, triangles to $p = 1.25$, crosses to $p = 1.5$, stars to $p = 1.75$ and squares to $p = 2$. Note that the discontinuity at $\alpha = 1$ disappears for the averaged power loss Δ_p/T.

increases for a given value of α, except from a certain value of α near $\sqrt{3}$. It is worthwhile to mention here the importance of the system on the initial conditions, since it has multiple equilibria. Numerical computations for higher values of the damping exponent p show similar behavior as the one described here.

3. Energy dissipation as a function of time

The results we have described previously are not easily obtained from the temporal evolution of the amplitude of the oscillation $x(t)$ in Fig. 1. For example, at first glance, one would say that the energy dissipation for $p = 0$ and $\alpha = 0.5$ (Fig. 1(a)) is stronger than for $p = 0$ and $\alpha = 5$ (Fig. 1(e)), whereas our results show that it is just the opposite. To show that our computations make sense, we have displayed in Fig. 6 some numerical estimates of the total energy of the system $E(t)$ for several values of p and α. It is clear that the behavior of $E(t)$ as a function of p and α follows that encountered for Δ_p, despite the fact that the solutions for the unforced and undamped Duffing oscillator used in the computation of Δ_p are quite different from the true trajectories shown in Fig. 1. In any case, it should be noticed that Δ_p represents in fact the loss of energy in the first cycle of the orbit, and that this dissipation of energy should be different in successive cycles. For example, in Fig. 6(c), which corresponds to $\alpha = 2$, the strongest energy loss corresponds to $p = 0$ for long values of the time variable t, whereas Δ_p is bigger for $p = 3$ than for $p = 0$, which coincides with the behavior of $E(t)$ for small values of t.

Given a dynamical system, the divergence theorem relates the volume contraction in phase space with the divergence of the flow f in the following manner [5]

$$\frac{\dot{V}(t)}{V(t)} = \operatorname{div} f. \tag{6}$$

In the linear damping case $\operatorname{div} f = -b$ and hence $V(t) = V_0\,\mathrm{e}^{-bt}$, which shows an exponential contraction of the volume in phase space. In a general case, for our model of nonlinear damping, we have

$$\frac{\dot{V}(t)}{V(t)} = -bp|\dot{x}|^{p-1}, \tag{7}$$

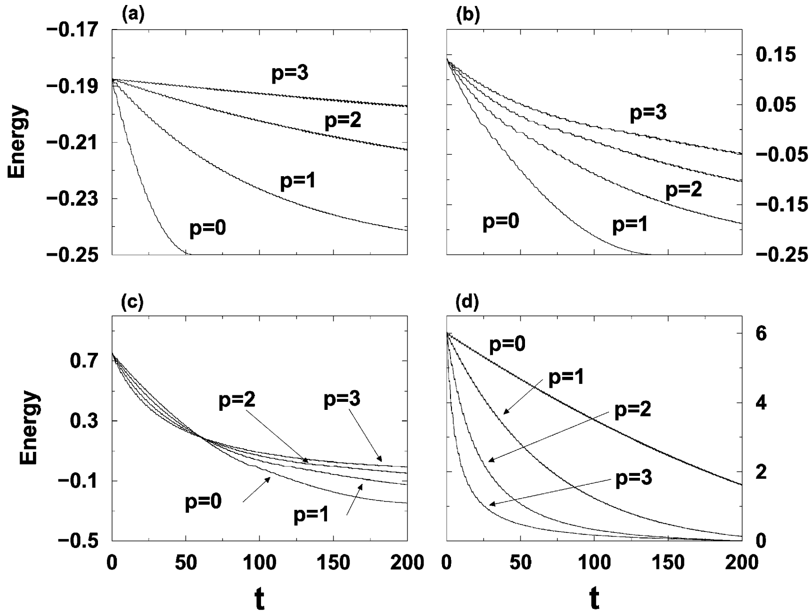

Fig. 6. Total energy of the nonlinearly damped Duffing oscillator versus time for different values of p and $\alpha = \sqrt{1+4E}$. Note that the behavior of $E(t)$ follows that of Δ_p when t is not too large, as is explained in the text: (a) $\alpha = 0.5$, (b) $\alpha = 1.25$, (c) $\alpha = 2$ and (d) $\alpha = 5$.

so the contraction is not exponential (note that this expression does not apply for the case $p = 0$, since the flow is discontinuous at $\dot{x} = 0$). The previous discussion is somehow related to the idea of the logarithmic decrement, which is a useful insight on the manner in which energy dissipates, defined as the natural logarithm of the ratio of the amplitudes of the oscillation on successive cycles [4] by

$$\delta_n = \ln\left(\frac{x(t) + (n-1)T}{x(t+nT)}\right). \quad (8)$$

In this expression, $x(t)$ is the amplitude of the oscillation in the time t, T the period and the index n indicates the successive cycles. In the general case, δ_n depends on the damping exponent p and the initial energy α. However, it is well known that in the case of a harmonic oscillator with viscous damping, where the above quantity usually appears, δ_n depends only on the damping coefficient (b in Eq. (2)), but not on n or on the initial energy.

As a method to characterize the way in which the nonlinear oscillator dissipates energy we have numerically explored the behavior of δ_n, since it gives us information about the decay of the oscillations for any general damping term. In Fig. 7, we have depicted δ_n versus n for several values of p and α. This figure shows that the logarithmic decrement depends on n (i.e., it depends on the pair of successive cycles considered) and also on the initial energy, α, and the damping exponent p. We can analyze this dependence from different points of view. A result of our analysis is that, when p is fixed, δ_1 depends on α just in the way that one would expect from the behavior of Δ_p, and the same can be said when α is fixed and p changes. Moreover, by the observation of the figures it can be deduced that this analysis can be extended to δ_n when n is small enough (typically lower than 5), despite Δ_p only concerns to the first cycle ($n = 1$). Another interesting result is that δ_n is an increasing function of n for all α when $p = 0$, and a decreasing function of n for all α when $p = 2, 3$, but it changes from a decreasing to an increasing function of n if $p = 1$ when α is varied. In fact, this implies that Δ_p passes through a maximum when considered as a

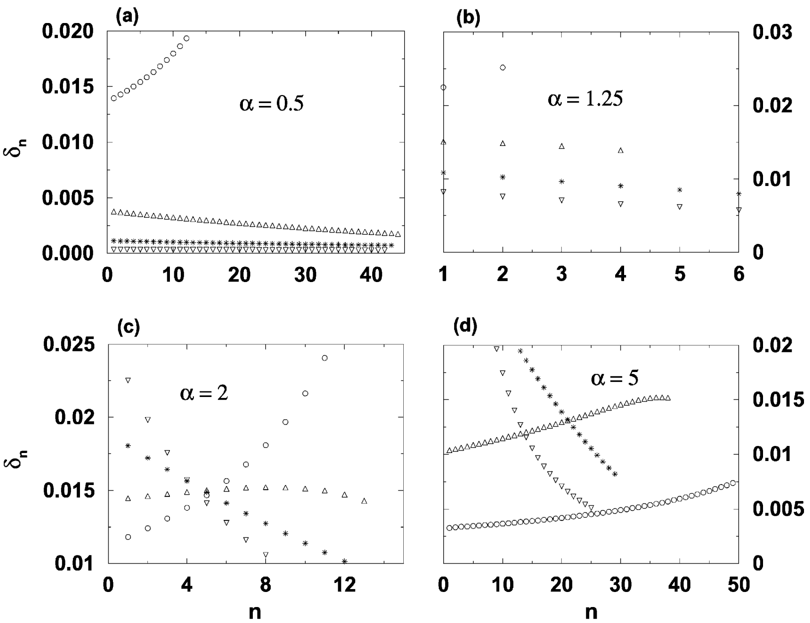

Fig. 7. Logarithmic decrement for successive cycles for different values of $\alpha = \sqrt{1+4E}$ and p. The dependence on this variable is discussed in the text. The symbols have the same meaning as in Fig. 2, i.e., circles stand for $p = 0$, triangles for $p = 1$, stars for $p = 2$ and inverted triangles for $p = 3$.

function of the logarithmic decrement in the case of viscous damping, exactly as in the case of a harmonic oscillator [4], whereas this is not true when $p \neq 1$.

4. Power dissipation and scaling law

The definition of the energy dissipation over a cycle that we have used (Eq. (5)) shows clearly that Δ_p explicitly depends on the period T in such a way that the energy dissipation over a cycle increases as far as the period increases. This suggests that it would be convenient to make use of a quantity independent of the duration of the cycle. For this reason, we define the *averaged power loss* in a period, Δ_p/T.

We have computed this magnitude for several values of the damping exponent p, and the results of our computations are plotted in Fig. 5(c) and (d). Analogous to Fig. 5(a) and (b), Fig. 5(c) shows the curves for p varying from 0 to 1 at intervals of 0.25, whereas Fig. 5(d) shows the curves for p values varying in the same way but between 1 and 2, both included. Perhaps the most interesting result related to the averaged power loss is the evidence that the discontinuity at $\alpha = 1$ present in Fig. 2 for Δ_p seems to disappear here, in other words, what we have termed averaged power loss looks like a continuous function of α. We have studied the behavior of Δ_p/T for different values of the damping exponent p (analytically for integer values of p and numerically for noninteger ones), having found that Δ_p/T is in fact a continuous function of α. It should be noticed as well the presence of local maxima in the curves for both cases $\alpha < 1$ and $\alpha > 1$. In the first case, the maximum location does not change when p varies, but it does in the case $\alpha > 1$ (see Fig. 5(c), where two maxima are shown for $\alpha > 1$).

As we mentioned in Section 2, the observation of Fig. 2 related to the energy dissipation Δ_p versus the energy α implies that the gap that appears at the discontinuity $\alpha = 1$ varies for the different damping exponents p. This suggest that a certain kind of scaling law should exist for $\lambda(p)$, where $\lambda(p)$ represents the gap in the discontinuity at $\alpha = 1$ for the Δ_p curves as a function of p. We have computed $\lambda(p)$ numerically for different values of p varying from $p = 0$ to 15 (p being an integer, see Fig. 8). With the sequence we have obtained, we have applied the Aitken's method

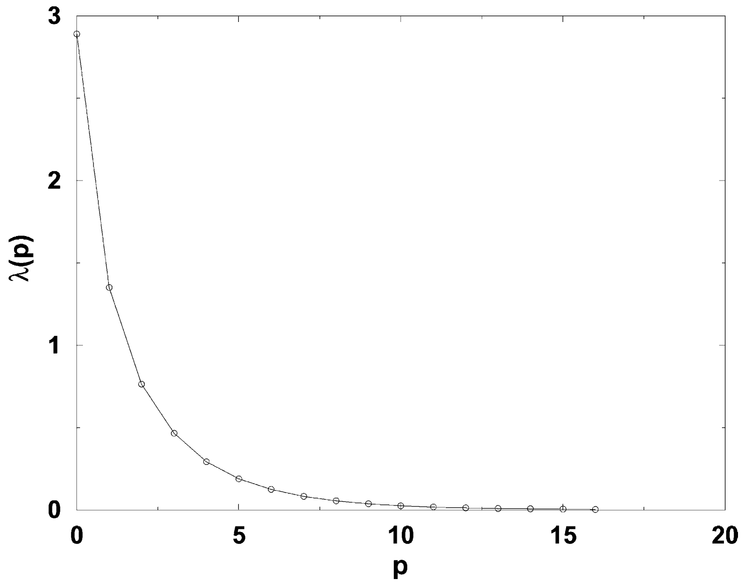

Fig. 8. Value of the discontinuity of Δ_p at $\alpha = 1$ for several integer values of the damping exponents p. Symbols correspond to the numerical computations. The solid line joining the symbols is plotted as a simple visualization aid, and has no physical meaning.

of quick convergence [20] in order to get evidence of such a scaling law. In order to apply this method to our case, we build a new sequence $d_p = \lambda(p-1) - \lambda(p)$ from the sequence of $\lambda(p)$, and afterwards define the quantity $\mu_p = d_p/d_{p+1}$. Then we analyze how μ_p behaves for increasing values of p. If μ_p tends to a constant, then a scaling law exists for $\lambda(p)$. We have found numerically that such a constant value exists and it is close to 1.48. This scaling law can also be calculated using a semi-analytical approach. If the points $(p, \lambda(p))$ are depicted in a semi-logarithmic scale, we see that the last ones (from $p = 8$ to 15) can be fitted to a straight line, i.e., they obey a decaying exponential law of the type $\lambda(p) = \lambda_0 \exp(-kp)$, where in our case, $\lambda_0 = 1.24117$ and $k = 0.39102$. Using this expression to calculate μ_p, one obtains $\mu_p = \mathrm{e}^k = 1.478$ for all p, which is very close to the result obtained when the discrete sequence is used.

5. Conclusions

We have thoroughly analyzed the behavior of a nonlinearly damped Duffing oscillator with a dissipative term proportional to the power of the velocity. The energy dissipation over a cycle for different damping terms, such as dry, linear, quadratic and cubic friction, has been analytically calculated, providing explicit formulae for these cases. These formulae explicitly show the dependence of the energy dissipation on the initial energy α and on the damping exponent p. Numerical computations have also been performed for noninteger powers in the damping term, resulting in a dependence very similar to the case of integer exponent.

The presence of a discontinuity at the value of $\alpha = 1$ and an inversion of the behavior of Δ_p for values of the energy above $\alpha \simeq \sqrt{3}$ have been analyzed and explained in terms of the dynamical behavior of the model. These results are contrasted with the evolution of the energy dissipation as a function of time for different values of the damping exponent p and initial energy α, finding that the dependence on these parameters for small values of time t is coherent with the results found for the dissipation of the energy Δ_p. In order to avoid the dependence of the energy dissipation on the period, we have defined an averaged power loss in a period. The information provided by this new magnitude is rather similar to the one provided by the energy dissipation over a cycle, although no discontinuity is present in this case.

As a usual tool in the study of the decay of the oscillations, we have also computed the logarithmic decrement for our system for different values of p and α, having obtained consistence with our previous results and some resemblances with the harmonic oscillator when $p = 1$.

Finally, we have analyzed the discontinuity gap which appears for the energy dissipation at $\alpha = 1$, and we have obtained numerically that this gap decreases as the power of the damping term increases. The analysis of the sequence of gaps for different damping exponents p has shown the presence of a scaling law associated with this dependence.

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Appendix A. Analytical solutions of the Duffing oscillator

We want to explicitly obtain the exact solutions for the double-well Duffing oscillator

$$\ddot{x} - x + x^3 = 0. \tag{A.1}$$

Our objective is to look for analytical expressions for the solutions and the period of the motions of the system. The double-well potential $V(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2$ is shown in Fig. 3, and we have two different kinds of motion depending on the energy of the orbit. There is a motion inside a well, for values $-\frac{1}{4} < E < 0$, where the solution of the period is given by

$$T(\alpha) = \frac{2\sqrt{2}}{\sqrt{1+\alpha}} K\left(\sqrt{\frac{2\alpha}{1+\alpha}}\right), \tag{A.2}$$

where $K(m)$ is the *complete elliptic integral of first kind* [18,19], and m is the elliptic modulus. Inside one of the small well $-\frac{1}{4} < E < 0$ and consequently $0 < \alpha < 1$, and the elliptic modulus m satisfies the condition $0 < m < 1$, as expected. Fig. 4 shows the explicit dependence of the period of the Duffing oscillator on the energy.

The exact solution for an orbit in this region is given by

$$x(t) = \sqrt{\alpha+1}\,\mathrm{dn}\left(\sqrt{\frac{\alpha+1}{2}}t, \sqrt{\frac{2\alpha}{\alpha+1}}\right), \qquad \text{(A.3)}$$

where $\alpha = \sqrt{1+4E}$ and $-\frac{1}{4} < E < 0$.

The period of an orbit for energies $E > 0$ is given by

$$T(\alpha) = \frac{4}{\sqrt{\alpha}} K\left(\sqrt{\frac{1+\alpha}{2\alpha}}\right), \qquad \text{(A.4)}$$

where as $E > 0$, then $\alpha > 1$, and consequently m satisfies the condition $1/\sqrt{2} < m < 1$. The exact solution for an orbit in this region is given by

$$x(t) = \sqrt{\alpha+1}\,\mathrm{cn}\left(\sqrt{\alpha}t, \sqrt{\frac{\alpha+1}{2\alpha}}\right), \qquad \text{(A.5)}$$

where $\alpha = \sqrt{1+4E}$ and $E > 0$.

Appendix B. List of integrals involving elliptic functions

We include a list of integrals that have been used in this work, based in [18,19]. In the list, $\mathrm{sn}(x,m)$, $\mathrm{cn}(x,m)$ and $\mathrm{dn}(x,m)$ are the Jacobian elliptic functions, and $K(m)$ and $E(m)$ are the complete elliptic integrals of the first and second kind, respectively.

$$\int_0^{4K(m)} \mathrm{dn}^2(x,m)\,\mathrm{d}x = 4E(m), \qquad \text{(B.1)}$$

$$\int_0^{4K(m)} \mathrm{dn}^4(x,m)\,\mathrm{d}x = \tfrac{4}{3}((m^2-1)K(m) + 2(2-m^2)E(m)), \qquad \text{(B.2)}$$

$$\int_0^{4K(m)} \mathrm{dn}^6(x,m)\,\mathrm{d}x = \tfrac{4}{15}(4(2-m^2)(m^2-1)K(m) + (8m^4-23m^2+23)E(m)), \qquad \text{(B.3)}$$

$$\int_0^{4K(m)} \mathrm{dn}^8(x,m)\,\mathrm{d}x = \tfrac{4}{7}\left(\tfrac{5}{3}(1-m^2) - \tfrac{8}{5}(2-m^2)^2\right)(1-m^2)K(m) + \tfrac{4}{7}\left(\tfrac{2}{5}(8m^4-23m^2+23) - \tfrac{10}{3}(1-m^2)\right) \times (2-m^2)E(m), \qquad \text{(B.4)}$$

$$\int \mathrm{sn}(x,m)\,\mathrm{cn}(x,m)\,\mathrm{d}x = -\frac{1}{m^2}\mathrm{dn}(x,m), \qquad \text{(B.5)}$$

$$\int \mathrm{sn}(x,m)\,\mathrm{dn}(x,m)\,\mathrm{d}x = -\mathrm{cn}(x,m). \qquad \text{(B.6)}$$

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