Vibrational resonance in a noise-induced structure

A. A. Zaikin,[1] L. López,[2] J. P. Baltanás,[2] J. Kurths,[1] and M. A. F. Sanjuán[2]

[1]*Institute of Physics, University of Potsdam, Am Neuen Palais 10, 14469 Potsdam, Germany*
[2]*Nonlinear Dynamics and Chaos Group, Universidad Rey Juan Carlos, 28933 Móstoles, Madrid, Spain*
(Received 26 March 2002; published 22 July 2002)

We report on the effect of vibrational resonance in a spatially extended system of coupled noisy oscillators under the action of two periodic forces, a low-frequency one (signal) and a high-frequency one (carrier). Vibrational resonance manifests itself in the fact that for optimally selected values of high-frequency force amplitude, the response of the system to a low-frequency signal is optimal. This phenomenon is a synthesis of two effects, a noise-induced phase transition leading to bistability, and a conventional vibrational resonance, resulting in the optimization of signal processing. Numerical simulations, which demonstrate this effect for an extended system, can be understood by means of a zero-dimensional "effective" model. The behavior of this "effective" model is also confirmed by an experimental realization of an electronic circuit.

The most counterintuitive property of white noise, its ability to induce ordering in nonlinear nonequilibrium systems, has been demonstrated in the effect of stochastic resonance (SR). In the usual situation a signal and additive noise are acting upon a system with a bistable potential. The dependence of the response on the additive noise intensity has a resonant form, hence an optimal value for signal processing noise intensity can be found [1,2]. This effect has proven to be general and has been found in many natural systems [3]. Among other systems showing this phenomenon we mention systems with excitable dynamics [4], systems without an external force [5] and systems without any kind of threshold [6].

It has been pointed out that SR-like phenomena can be also observed in systems where a chaotic signal is used instead of noise [7]. Moreover, in [8] it has been shown that a high-frequency periodic force can work as a noise and amplify the response to the low-frequency periodic signal in bistable systems. This effect has been called vibrational resonance (VR) [9], analogously to SR. In VR the dependence of the system response versus the amplitude of the high-frequency action has a well-known bell-shaped resonant form. Since two-frequency signals are very often used in communication technologies [10], it means that an optimal high-frequency modulation may improve processing of a low-frequency signal. It is important to mention that two-frequency signals are also object of intensive interest in laser physics [11], acoustics [12], neuroscience [13], or physics of the ionosphere [14]. The aim of this paper is to investigate whether VR can be achieved in noise-induced structures, which do not have any threshold or a potential barrier in the absence of noise. For this purpose we consider a spatially extended system consisting of a network of coupled monostable noisy oscillators under the action of low- and high-frequency periodic signals. In this system a collective action of coupling and multiplicative noise results in the organization of bistability of the mean field. If the amplitude of a low-frequency signal is not enough for a synchronous response of the system, then the high-frequency force is applied. We find that an increase of the high-frequency amplitude leads to a nonmonotonous change of the system

response with a clearly defined maximum. Therefore, we present a different phenomenon, *vibrational resonance in a noise-induced structure*, which is a variation of SR.

We study this effect on a nonlinear lattice of coupled overdamped oscillators introduced in [15] and further studied in [16,17]. The following set of Langevin equations describes the considered system:

$$\dot{x}_i = f(x_i) + g(x_i)\xi_i(t) + \frac{D}{2d} \sum_{j \in nn(i)} (x_j - x_i) + A\cos(\omega t) + B\cos(\Omega t), \tag{1}$$

where $x_i(t)$ represents the state of the ith oscillator, $i = 1, \ldots, L^d$, in the cubic lattice of size L in d dimensions with $N = L^d$ elements. The sum runs over the nearest neighbors of the ith cell $[nn(i)]$, and the strength of the coupling is measured by D. The noisy term $\xi_i(t)$ represents Gaussian noise with zero mean and is uncorrelated both in space and time,

$$\langle \xi_i(t)\xi_j(t') \rangle = \sigma_m^2 \delta_{i,j}\delta(t-t'). \tag{2}$$

The last terms in Eq. (1) stand for external periodic forces, representing a low-frequency signal with amplitude A, frequency ω, and a high-frequency signal with amplitude B and frequency Ω, where $\Omega \gg \omega$ and these frequencies can be incommensurable.

For the sake of simplicity, the functions $f(x)$ and $g(x)$ are taken to be of the form [18]

$$f(x) = \begin{cases} -G_b x - (G_a - G_b)B_p & \text{if} \quad x \leq -B_p, \\ -G_a x & \text{if} \quad |x| < B_p, \\ -G_b x + (G_a - G_b)B_p & \text{if} \quad x \geq B_p, \end{cases} \tag{3}$$

$$g(x) = x, \tag{4}$$

where the parameters $G_a = 0.5$, $G_b = 10$, and $B_p = 1$ determine the slopes and the break point of the piecewise-linear characteristic curve [an approximation of the function $f(x)$

$=-x-x^3$]. Such forms of functions describe a realistic electronic circuit designed in [18].

In the absence of the external force ($A=0,B=0$) this model can be solved analytically by means of a standard mean-field theory procedure [19]. The mean-field approximation consists in replacing the nearest-neighbor interaction by a global term in the Fokker-Planck equation corresponding to Eq. (1). Using this mean-field approximation, one determines transitions between ordered ($m\neq 0$) and disordered ($m=0$) phases [18], where m is the mean field, defined as $m(t)=1/L^d\Sigma_{i=1}^N x_i(t)$. This analysis shows that the joint action of multiplicative noise and coupling between the elements leads to the bistability of the mean field (ordered phase). If we fix the coupling strength above its critical value, then an increase of the multiplicative noise induces a disorder-order phase transition, which is followed by a reentrant transition to disorder [15]. In the ordered phase the system occupies one of two possible symmetric states with the mean fields $m_1=-m_2\neq 0$, depending on the initial conditions. This bistability disappears if we switch off the multiplicative noise.

The mechanism of this effect can be understood as follows. In the absence of the external signal ($A=0,B=0$) the time evolution of the first moment of a single element is given simply by the drift part in the corresponding Fokker-Planck equation (Stratonovich case):

$$\langle \dot{x}_i \rangle = \langle f(x_i) \rangle + \frac{\sigma_m^2}{2}\langle g(x_i)g'(x_i)\rangle. \qquad (5)$$

Following the short time evolution approximation described in [16], suppose that we start with an initial Dirac δ probability density, and that we follow it only for a short time, such that fluctuations are small and the probability density is well approximated by a Gaussian. Then the equation for the maximum of the probability, which is also the average value $\bar{x}_i=\langle x_i\rangle$ in this approximation, takes the following form:

$$\dot{\bar{x}}_i = f(\bar{x}_i) + \frac{\sigma_m^2}{2}g(\bar{x}_i)g'(\bar{x}_i), \qquad (6)$$

which is valid if $f(\langle x_i\rangle)\gg\langle \delta x_i^2\rangle f''(\langle x_i\rangle)$. Returning to the extended system, in the mean-field approximation the field average $\langle x\rangle=\bar{x}$ is described by the first moment of any of the single elements given by Eq. (5). Furthermore, suppression of fluctuations, performed by coupling, makes this approximation appropriate along the whole time evolution [21]. Then we can write also for the field average

$$\dot{\bar{x}} = f(\bar{x}) + \frac{\sigma_m^2}{2}g(\bar{x})g'(\bar{x}), \qquad (7)$$

which again is valid if $f(\langle x\rangle)\gg\langle \delta x^2\rangle f''(\langle x\rangle)$ [20]. For this dynamics an "effective" potential $U_{\mathrm{eff}}(x)$ can be derived, which has the form

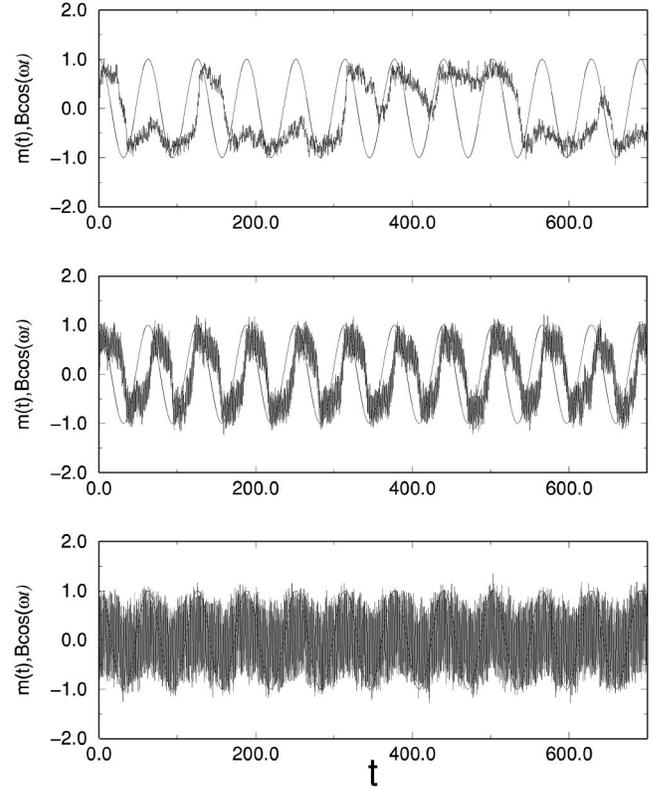$$U_{\mathrm{eff}}(x) = U_0(x) + U_{\mathrm{noise}} = -\int f(x)dx - \frac{\sigma_m^2 g^2(x)}{4}, \qquad (8)$$

FIG. 1. Time series of the mean field of the system [Eq. (1)] compared with the low-frequency signal $A\cos(\omega t)$ (not in scale) for different intensities of high-frequency vibration. From top to bottom, $B=0.5$, 1.5, and 4.0. $\Omega=5.0, \omega=0.1, A=0.15, \sigma_m^2=3.0$. This intensity of multiplicative noise corresponds to the bistable region.

where $U_0(x)$ is a monostable potential and U_{noise} represents the influence of the multiplicative noise. In the region where VR in the noise-induced structure is observed, this potential has a bistable form due to the input provided by multiplicative noise.

Now let us turn to the problem, i.e., how the system (1) responds to a periodic signal that contains two quite different frequencies (e.g., $\omega=0.1$ and $\Omega=5$). First, we analyze the behavior of the system in the parameter region (the parameters being the coupling strength and the multiplicative noise intensity), where the noise-induced bistability is provided. We set the amplitude of the low-frequency signal A fixed and sufficiently small (e.g., $A=0.15$), which is not enough to cause jumps between two potential wells. The time series of the mean field $m(t)$ and the corresponding periodic input signal are plotted in Fig. 1 for three different values of B (increasing from top to bottom). For a small amplitude B we observe rare jumps between the two symmetric states m_1 and m_2 in the output, which are not synchronized with the low-frequency signal (here $d=2$ and $N=10$). If we increase B to its optimal value (in the middle), it is clearly seen that hops occur with the same periodicity as the input signal. Hence, the high-frequency modulation optimizes signal processing in this noise-induced bistable structure. Further increase of B leads to oscillations hops at the high frequency, which completely hide the signal at the low frequency. The situation differs qualitatively when we choose another intensity of
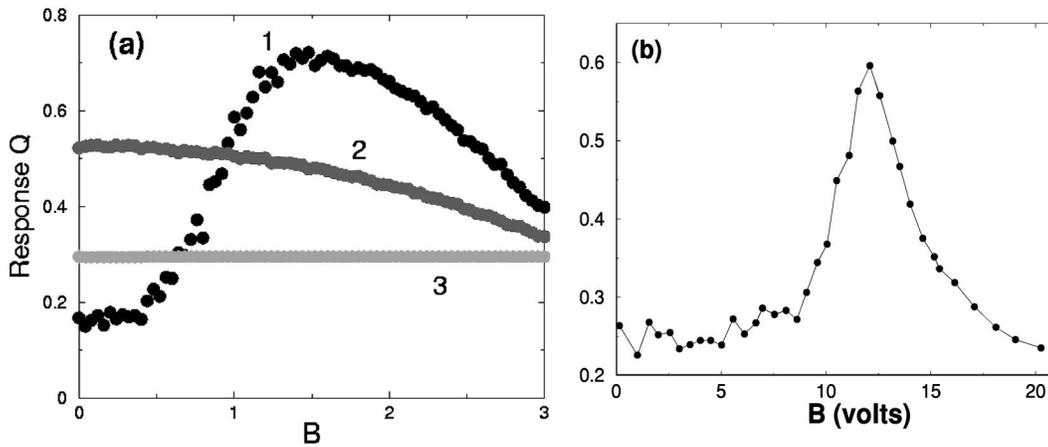
FIG. 2. Vibrational resonance in the noise-induced structure. Numerical simulations (a) vs experimental results for the effective model (b). Response Q of the system vs the amplitude of the high-frequency force. In (a) $\sigma_m^2 = 3$ (label 1), 0.5 (label 2), and 0 (label 3); other parameters are the same as in Fig. 2.

multiplicative noise corresponding to the monostable region. In this case, an increase of B leads only to the destruction of synchronization between input and output. Hence, the high-frequency modulation is unable to improve the quality of signal processing at low frequency in this case. Therefore, the system considered exhibits *vibrational resonance in a noise-induced structure* only when a collective bistability has been created by multiplicative noise and coupling.

To characterize this VR effect quantitatively, we calculate the dependence of the system response Q at the signal frequency on the amplitude of the high-frequency force [Fig. 2(a)]. For the bistable regime the response curve (label 1) exhibits a clearly defined maximum for the optimal value of B, which gives evidence for the presence of VR. Note that this effect disappears if we decrease (label 2) or switch off (label 3) the multiplicative noise: in this case an increase of the amplitude of a high-frequency force may lead only to the decrease of the system response.

When the intensity of the multiplicative noise corresponds to the bistable region of the mean field, this effect may be understood assuming a model of an overdamped system with a bistable potential [see Eq. (8)] under the action of a high- and a low-frequency periodic force,

$$\dot{m} = F(m) + A\cos(\omega t) + B\cos(\Omega t) + \xi(t), \qquad (9)$$

where $m(t)$ is the mean field of the initial system, and the function $F(x)$ describes a bistable potential. The noisy term $\xi(t)$ denotes tiny fluctuations, which are present in all real systems. We have constructed an electronic circuit (Fig. 3) in order to make analog simulations of Eq. (9). From these simulations, one obtains that the "effective" model undergoes the effect of VR [see Fig. 2(b)]; that is, for intermediate values of the high-frequency signal, the jumps of the particle are synchronized with the low-frequency signal, whereas this synchronization almost disappears for very high or very low values of the amplitudes of the high-frequency signal.

It is worth noting that not every system with noise-induced bistability exhibits vibrational resonance. For example, zero-dimensional systems, described in [22], demonstrate noise-induced bistability due to the bistability of a so-called "stochastic" potential but do not show a pronounced VR. Although it is possible to observe a small maximum in the response of the system, a further increase of the multiplicative noise, which provides bistability, decreases the response of the system.

In conclusion, we have described a phenomenon consisting in the existence of vibrational resonance in a noise-induced structure. This effect is a synthesis of a noise-induced phase transition and vibrational resonance. High-frequency carrier force is able to optimize signal processing, and this process
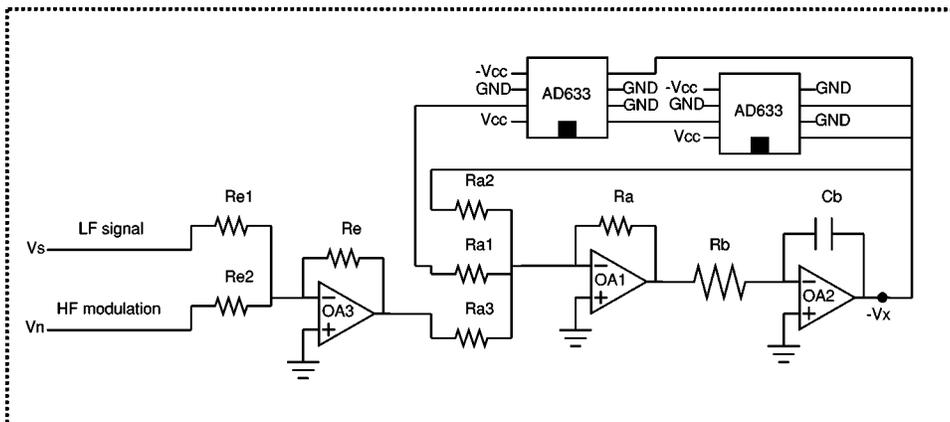

FIG. 3. Electronic circuit for the effective model [Eq. (9)]. OA are operational amplifiers, the R's read for resistors and the AD633's stand for two multiplyers with coefficient α. C_b is a capacitor.

can be controlled by multiplicative noise. Numerical simulations for a spatially extended system have been confirmed by experimental results for a zero-dimensional "effective" model. We expect that due to its generality, this effect can be of much importance in communication technologies.

We expect that these theoretical findings will stimulate experimental work in order to verify VR in noise-induced structures in real physical systems (for the first experimental observation of noise-induced bistability see [23]). Appropriate situations can be found in electronic circuits [18], electronic cellular neural networks [24], as well as in systems that show a noise-induced shift of the phase transition, e.g.,

in liquid crystals [25], photosensitive chemical reactions [26], or Rayleigh-Bénard convection [27]. We believe that the results shown in this paper might be crucial for such experiments because in the noise-induced structure presented here, the bistability of the mean field is controlled by noise.

[1] R. Benzi, A. Sutera, and A. Vulpiani, J. Phys. A **14**, L453 (1981).

[2] L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. **70**, 223 (1998).

[3] J. Douglass, L. Wilkens, and L. Pantazelou, Nature (London) **365**, 337 (1993).

[4] K. Wiesenfeld *et al.*, Phys. Rev. Lett. **72**, 2125 (1994).

[5] H. Gang, T. Ditzinger, C. Ning, and H. Haken, Phys. Rev. Lett. **71**, 807 (1993); A. Pikovksy and J. Kurths, *ibid.* **78**, 775 (1997).

[6] S. M. Bezrukov and I. Vodyanoy, Nature (London) **385**, 319 (1997).

[7] S. Sinha, Physica A **270**, 204 (1999).

[8] P. Landa and P. McClintock, J. Phys. A **33**, L433 (2000).

[9] See also the pioneering work about the maintenance of vibrations: J. Rayleigh, *The Theory of Sound* (Macmillan, London, 1894).

[10] V. Mironov and V. Sokolov, Radiotekh. Elektron. (Moscow) **41**, 1501 (1996), (in Russian).

[11] D. Su, M. Chiu, and C. Chen, Precis. Eng. **18**, 161 (1996).

[12] A. Maksimov, Ultrasonics **35**, 79 (1997).

[13] J. Victor and M. Conte, Visual Neurosci. **17**, 959 (2000).

[14] V. Gherm, N. Zernov, B. Lundborg, and A. Vastberg, J. Atmos. Sol.-Terr. Phys. **59**, 1831 (1997).

[15] C. Van den Broeck, J. M. R. Parrondo, and R. Toral, Phys. Rev. Lett. **73**, 3395 (1994).

[16] C. Van den Broeck, J. M. R. Parrondo, R. Toral, and R. Kawai, Phys. Rev. E **55**, 4084 (1997).

[17] A. Zaikin, J. Kurths, and L. Schimansky-Geier, Phys. Rev. Lett. **85**, 227 (2000).

[18] A. Zaikin, K. Murali, and J. Kurths, Phys. Rev. E **63**, 020103(R) (2001).

[19] J. García-Ojalvo and J. M. Sancho, *Noise in Spatially Extended Systems* (Springer, New York, 1999).

[20] Note that not always this approximation appropriately explains the phase transition in this system [A. A. Zaikin, J. García-Ojalvo, and L. Schimansky-Geier, Phys. Rev. E **60**, R6275 (1999)].

[21] C. Van den Broeck, in *Stochastic Dynamics*, edited by L. Schimansky-Geier and T. Pöschel (Springer, Heidelberg, 1997), p. 7.

[22] W. Horsthemke and R. Lefever, *Noise-Induced Transitions* (Springer, Berlin, 1984).

[23] D. Grisowld and J. Tough, Phys. Rev. A **36**, 1360 (1987).

[24] V. Pérez-Muñuzuri, V. Pérez-Villar, and Leon O. Chua, IEEE Trans. Circuits Syst., I: Fundam. Theory Appl. **40**, 174 (1993); P. Arena, R. Caponetto, L. Fortuna, and A. Rizzo, *ibid.* **48**, 360 (2001); M. Rabinovich, A. Volkovskii, P. Lecanda, R. Huerta, H. D. I. Abarbanel, and G. Laurent, Phys. Rev. Lett. **87**, 068102 (2001).

[25] S. Kai, T. Kai, and M. Takata, J. Phys. Soc. Jpn. **47**, 1379 (1979).

[26] J. Micheau, W. Horsthemke, and R. Lefever, J. Chem. Phys. **81**, 2450 (1984).

[27] C. Meyer, G. Ahlers, and D. Cannell, Phys. Rev. A **44**, 2514 (1991).