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# ADAPTIVE PROCEDURE FOR THE PARAMETER ESTIMATION OF A MODEL OF A $CO_2$ CHAOTIC LASER

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We have proposed an adaptive procedure in order to estimate unknown parameters of a numerical model of a chaotic $CO_2$ laser, which is observed through a time series that represents the output intensity of the laser. To do that, we consider a coupled system with the same functional form and adjustable parameters. The salient feature of the proposed technique is that accurate parameter estimation and identical synchronization can be jointly achieved by adaptively adjusting the desired parameters of the coupled system.

*Keywords*: Parameter estimation; chaos synchronization; chaotic laser.

## 1. Introduction

Nonlinear systems can exhibit very rich dynamical behavior, such as chaos, self-oscillations, bifurcations, switching, etc. An important issue in this type of systems is the estimation of the model parameters using scalar measurements from the system [Maybhate & Amritkar, 1999]. One typical approach to this problem consists the iterative processing of the whole set of available measurements. This includes several multiple-shooting techniques [Ghosh *et al.*, 2001] and statistical and Monte Carlo procedures [Pisarenko & Sornette, 2004; Sakaguchi, 2002]. However, iterative processing is computationally expensive and inadequate for problems where the observations must be handled online. An appealing strategy in this situation is to exploit the synchronization properties of coupled chaotic systems in order to attain parameter estimation [Parlitz, 1996; Maybhate & Amritkar, 1999].

In this work, we address the problem of estimating the unknown parameters of a numerical model that represents a chaotic laser. This is very useful in order to predict the dynamical behavior and to study the fundamental properties of this type of physical systems, which have important applications in different fields such as biology, medicine or engineering [VanWiggeren & Roy, 1998].

The remainder of this paper is organized as follows. In Sec. 2 we describe the numerical model of the chaotic $CO_2$ laser that we investigate. In Sec. 3 we propose a parameter estimation procedure based on the methodology of [Mariño & Míguez, 2006] in order to estimate scalar parameters of the laser model. We illustrate the validity of our technique with computer simulation results in Sec. 4. Finally, Sec. 5 is devoted to the conclusions.

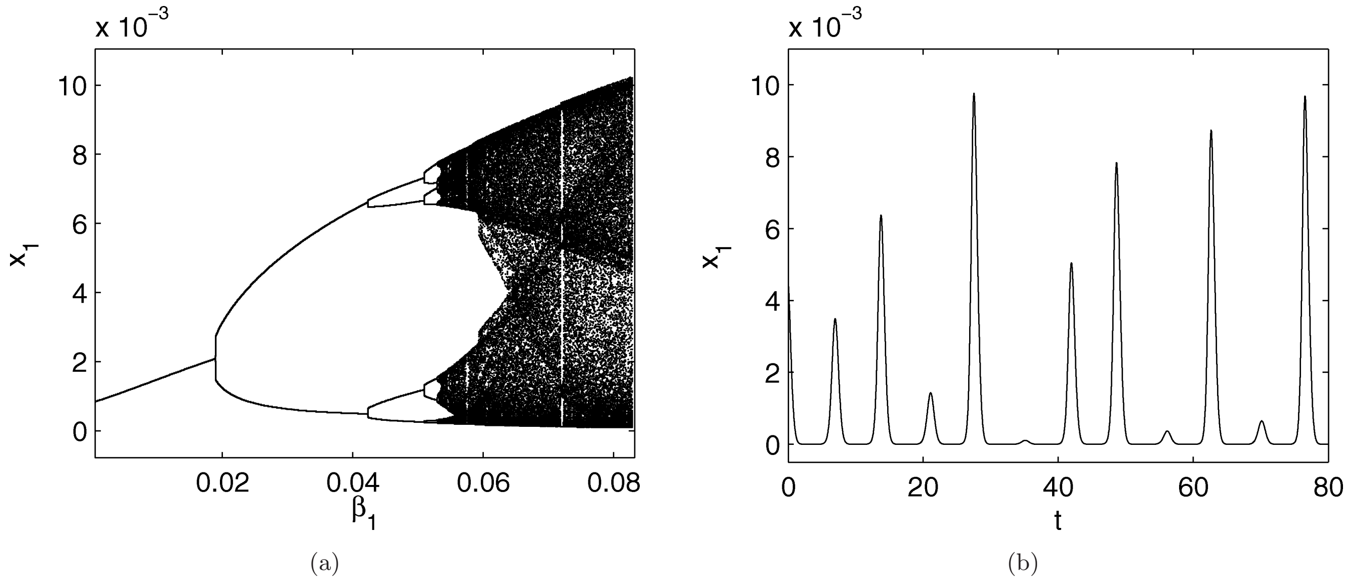



Fig. 1. (a) Bifurcation diagram for the $CO_2$ laser model as a function of the amplitude of the external forcing, $\beta_1$. (b) Temporal evolution of the laser intensity, $x_1$, for $\beta_1 = 0.08$.

## 2. $CO_2$ Laser Model

We consider the following model of five differential equations that represents the behavior of a $CO_2$ laser model [Mariño *et al.*, 2004], that is,

$$\begin{aligned} \dot{x}_1 &= kx_1(x_2 - 1 - \alpha \sin^2(F_1(t)) \\ \dot{x}_2 &= -\gamma_1 x_2 - 2kx_1x_2 + gx_3 + x_4 + p \\ \dot{x}_3 &= -\gamma_1 x_3 + gx_2 + x_5 + p \\ \dot{x}_4 &= -\gamma_2 x_4 + zx_2 + gx_5 + zp \\ \dot{x}_5 &= -\gamma_2 x_5 + zx_3 + gx_4 + zp, \end{aligned} \tag{1}$$

where

$$F_1(t) = \beta_1 \sin(2\pi ft) + b \tag{2}$$

is the external forcing signal of the laser. In the above equations, $x_1$ represents the laser output intensity, $x_2$ is the population inversion between the two resonant levels, and $x_3$, $x_4$ and $x_5$ account for molecular exchanges between the two levels resonant with the radiation field and the other rotational levels of the same vibrational band. The parameters of the model are the following: $k$ is the unperturbed cavity loss parameter, $g$ is a coupling constant, $\gamma_1$ and $\gamma_2$ are population relaxation rates, $z$ accounts for an effective number of rotational levels, $\alpha$ accounts for the efficiency of the electro-optic modulator and $p$ is the pump parameter. The rest of the parameters are related to the external periodic forcing. In particular, $f$ is the frequency, $b$ is the bias voltage and $\beta_1$ is the amplitude of the external forcing.

We choose the following fixed values of the parameters: $k = 30$, $\alpha = 4$, $\gamma_1 = 10.0643$, $g = 0.05$, $p = 0.0198$, $\gamma_2 = 1.0643$, $z = 10$, $f = 1/7$, $b = 0.2$ and $\beta_1 = 0.08$, which corresponds to a chaotic regime. This can be seen in Figs. 1(a) and 1(b), which represent the characteristic bifurcation diagram for the $CO_2$ laser model as a function of the parameter $\beta_1$ and the temporal evolution of the laser intensity, $x_1$, for $\beta_1 = 0.08$, respectively.

## 3. Parameter Estimation

We consider the parameter estimation methodology proposed in [Mariño & Míguez, 2006], which is based on the synchronization phenomenon that appears in coupled chaotic systems. We consider the laser output intensity $x_1$ as the only signal observed from the system modeled by Eq. (1), which we will subsequently refer to as *primary* system. It is assumed that the exact value of parameter $\beta_1$ in Eq. (2) is unknown and we are interested in estimating it. To do that, we consider a *secondary* laser, modeled by the same differential equations as the primary one, but with its external sinusoidal forcing function adequately modified. In particular, we consider

$$F_2(t) = \beta_2(1 + \epsilon(x_1 - y_1)) \sin(2\pi ft) + b, \tag{3}$$

where $y_1$ and $x_1$ represent the output intensity of the secondary and primary lasers, respectively, $\epsilon$ represents the coupling strength between the two systems, and $\beta_2$ is an adjustable parameter. It is known that for an appropriate value of the coupling strength, $\epsilon$, both systems synchronize when they have identical parameters values [Mariño *et al.*, 2004].

The approach to parameter estimation which we are going to consider, consists of a gradient-descent optimization of an adequate series of cost functions. In our case, we propose to use the following cost functions

$$J_n = |e(nT)|^2, \quad n = 1, 2, \ldots, \tag{4}$$

where $T$ is the period of the spikes of the laser intensity and $e(nT)$ is an error signal consisting of the difference between the first temporal derivatives of $x_1$ and $y_1$, i.e. $e(nT) = \dot{x}_1(nT) - \dot{y}_1(nT)$. Notice that the functions $J_n$ depend on $\beta_2$ through $\dot{y}_1(nT)$ and all of them attain a common minimum value at $\beta_2 = \beta_1$. In order to find this minimum of the series of cost functions, we assume that we have the ability to update the parameter $\beta_2$ every $T$ seconds. Therefore, we can compute the sequence of parameter estimates

$$\beta_{2,n} = \arg\min_{\beta_2}\{J_n\}, \quad n = 1, 2, \ldots. \tag{5}$$

A simple procedure to find the minimum of this series is to use the gradient-descent method

$$\beta_{2,n} = \beta_{2,n-1} - \mu\frac{\partial J_n}{\partial \beta_2}, \tag{6}$$

where $\mu$ is a step-size parameter.[1]

The problem of calculating $\partial J_n/\partial\beta_2$ reduces to finding an expression for $\partial|e(nT)|^2/\partial\beta_2$. It is straightforward to obtain that

$$\frac{\partial|e|^2}{\partial\beta_2} = -2(\dot{x}_1 - \dot{y}_1)\frac{\partial\dot{y}_1}{\partial\beta_2}. \tag{7}$$

but, unfortunately, the derivative with respect to $\beta_2$ on the right-hand side of (7) cannot be expressed in closed form because of the complex implicit dependence of the dynamic variables on the parameters.[2] However, if we consider only the *explicit* derivative, Eq. (7) becomes very simple. Specifically, if we use notation $\partial'/\partial$ to denote explicit derivation (meaning that implicit dependencies of the variables on the parameters are neglected), (7) reduces to

$$\begin{aligned}\frac{\partial|e|^2}{\partial\beta_2} &\approx \frac{\partial'|e|^2}{\partial\beta_2}\\ &= 4(\dot{x}_1 - \dot{y}_1)kx_1\sin(F_2(t))\\ &\quad\times\cos(F_2(t))\frac{F_2(t)}{\beta_2}\end{aligned} \tag{8}$$

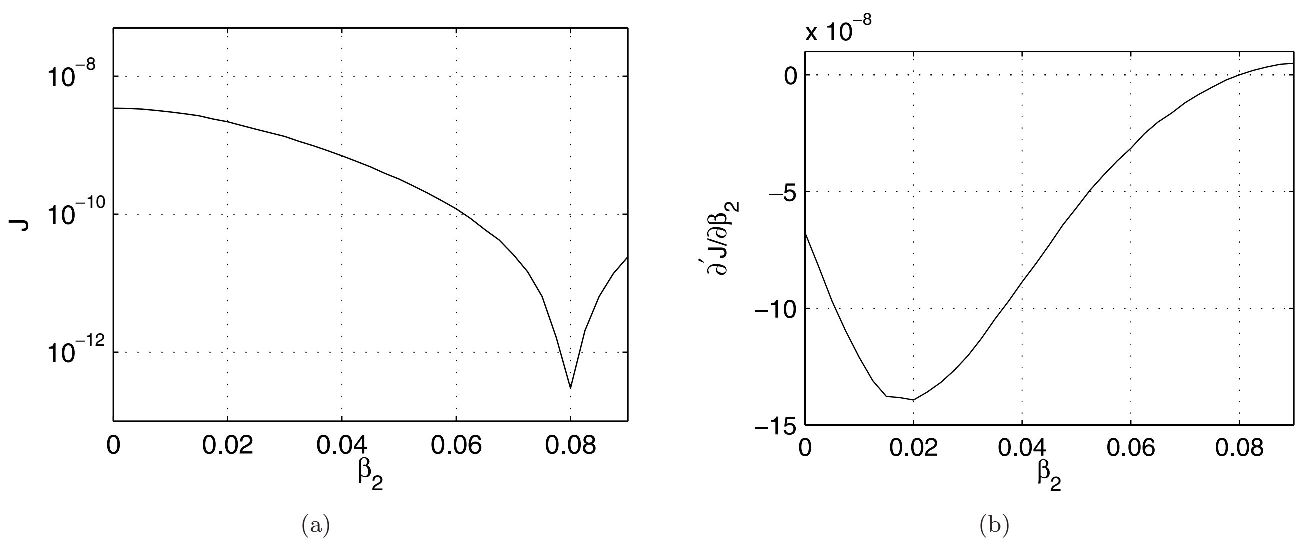
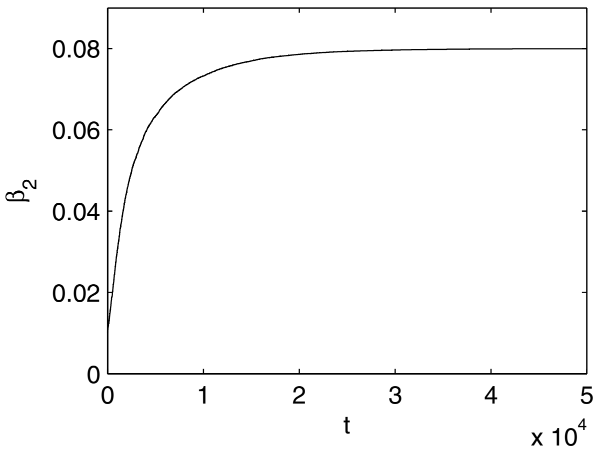


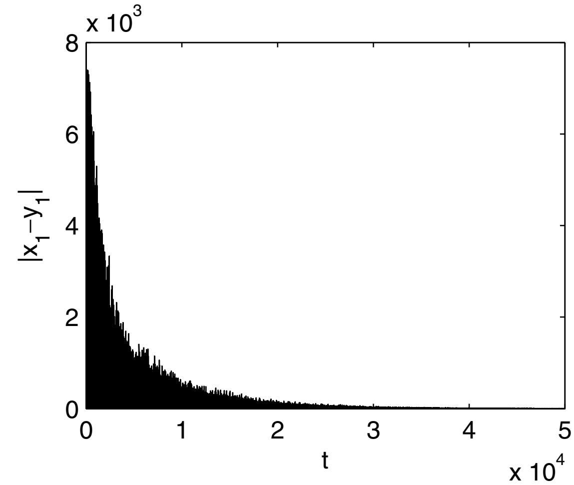

Fig. 2. (a) Temporal average, $J$, of the series of cost functions $J_n$ as a function of $\beta_2$ represented in a logarithmic scale. (b) Temporal average of the explicit derivative of the series of cost functions, labeled as $\partial' J/\partial\beta_2$, as a function of $\beta_2$.

[1] Other more sophisticated optimization methods could be used as described in [Hastie & Tibshirani, 1990].

[2] In a general case, if the available data are noisy, there may be errors in the numerical computation of instantaneous time derivatives. One obvious way of mitigating this problem is to use an appropriate filter to "clean" the observed signal before proceeding to compute its derivative (actually, such a filter will be a constituent part of any *good* differentiating circuit or system). Other refined methods to estimate derivatives can be found in [Ahnert & Abel, 2005].




(a)




(b)

Fig. 3. (a) Time evolution of the parameter $\beta_2$. (b) Time evolution of $|y_1 - x_1|$.

that yields the desired approximate derivative of $\partial J_n/\partial\beta_2$.

In order to verify the appropriateness of the described approximations, numerical simulations have been carried out to approximate both $J_n$ and its explicit derivative $\partial'/\partial\beta_2$ as functions of the model parameter $\beta_2$. We have assigned to the primary laser model the parameter values indicated in Sec. 2, in particular, $\beta_1 = 0.08$, which corresponds to the chaotic behavior indicated in Fig. 1(b). The parameter values for the secondary laser model are identical to the primary one, except $\beta_2$ that has taken values from 0 to 0.085, corresponding to the bifurcation diagram of Fig. 1(a). Figure 2(a) represents a temporal average of $J_n$, labeled as $J$, as a function of $\beta_2$ in a logarithmic scale, where the minimum is located at 0.08, which corresponds to the value of $\beta_1$. The validity of the approximation by the explicit derivative is illustrated in Fig. 2(b). We can observe that this derivative is positive when the function $J$ increases, is negative when this function decreases and vanishes at the minimum of $J$. Because of this relationship between the explicit derivative and the true derivative, the generic gradient algorithm of Eq. (6) is particularized to

$$\beta_{2,n} = \beta_{2,n-1} - \mu\frac{\partial' J_n}{\partial\beta_2}. \tag{9}$$

## 4. Numerical Results

We have carried out additional computer simulations in order to numerically demonstrate the performance of algorithm (9) in terms of both parameter estimation and synchronization accuracy. The parameter values in the primary system are the same as in Sec. 2 and we use a fourth-order Runge–Kutta method with step $h = 10^{-2}$ time units (t.u.) to numerically integrate the systems. The starting value for the parameter estimate is $\beta_{2,0} = 0.01$, which corresponds to a nonchaotic state of the secondary laser model (see the bifurcation diagram of Fig. 1(a)). The parameter adaptation period is set to $T = 700h = 7$ t.u.

Figure 3(a) shows the time evolution of the parameter $\beta_2$, that is, the convergence of the parameter estimate, and Fig. 3(b) shows the time evolution of the absolute deviation between the output intensity of the primary and secondary lasers, specifically, $|y_1 - x_1|$. It is clearly seen that identical synchronization of both laser intensities is achieved at the same pace as the parameter estimate converge to the desired value, that is $\beta_2 = 0.08$. As this occurs, the secondary laser, that begins with $\beta_{2,0} = 0.01$, is going through the different behaviors of the characteristic bifurcation diagram.

## 5. Conclusions

We have proposed an adaptive procedure in order to estimate unknown parameters of a numerical model of a chaotic $CO_2$ laser which is observed through a time series that represents the output intensity of the laser. To do that, we consider a coupled system with the same functional form and adjustable parameters. The salient feature of the proposed technique is that accurate parameter estimation and identical synchronization can be jointly achieved by adaptively adjusting the desired parameters of the coupled system. Here, we have shown how it is possible to estimate the amplitude of the external forcing signal that regulates the behavior of the laser, that is, parameter $\beta$, but estimation of other parameters, like the bias voltage of the external forcing signal, $b$, or the pump of the laser, $p$, have also been carried out with satisfactory results.

Finally, although the objective of this paper is to demonstrate the estimation method in a numerical model, it is interesting to discuss its implementation in real experiments. On the one hand, if an accurate model of the physical system is available, the latter can be regarded as the primary system and the secondary system be a numerical one, built from the model. In this case, it is necessary that the model yield a good representation of the experimental data. On the other hand, we can also consider that both the primary and the secondary systems are physical devices (with some adjustable parameter in the latter). Since the method is based on identical synchronization, it is important that, in this case, the systems be as tightly matched as possible, except for the parameter to be estimated. With this setup, the derivative of the cost function should be approximated numerically, instead of the analytical procedure described in this paper.

## Acknowledgments

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## References

Ahnert, K. & Abel, M. [2005] "Numerical differentiation: Local versus global methods," http://arxiv.org/abs/physics/0510176 and the references therein.

Ghosh, A., Ravi Kumar, V. & Kulkarni, B. D. [2001] "Parameter estimation in spatially extended systems: The Karhunen–Lóeve and Galerkin multiple shooting approach," *Phys. Rev. E* **64**, 056222.

Hastie, T. J. & Tibshirani, R. J. [1990] *Generalized Additive Models* (Chapman and Hall, NY).

Mariño, I. P., Allaria, E., Sanjuán, M. A. F., Meucci, R. & Arecchi, F. T. [2004] "Coupling scheme for complete synchronization of periodically forced chaotic $CO_2$ lasers," *Phys. Rev. E* **70**, 036208.

Mariño, I. P. & Míguez, J. [2006] "An approximate gradient-descent method for joint parameter estimation and synchronization of coupled chaotic systems," *Phys. Lett. A* **351**, 262–267.

Maybhate, A. & Amritkar, R. E. [1999] "Use of synchronization and adaptive control in parameter estimation from a time series," *Phys. Rev. E* **59**, 284–293.

Parlitz, U. [1996] "Estimating model parameters from time series by autosynchronization," *Phys. Rev. Lett.* **76**, 1232–1235.

Pisarenko, V. F. & Sornette, D. [2004] "Statistical methods of parameter estimation for deterministically chaotic time series," *Phys. Rev. E* **69**, 036122.

Sakaguchi, H. [2002] "Parameter evaluation from time sequences using chaos synchronization," *Phys. Rev. E* **65**, 027201.

VanWiggeren, G. D. & Roy, R. [1998] "Communication with chaotic lasers," *Science* **279**, 1198–1200.