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Control of Transient Chaos Using Safe Sets in Simple Dynamical Systems

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Abstract. Transient chaos is nearly as ubiquitous as chaos itself, and it is a manifestation of the existence of a nonattractive chaotic set: a chaotic saddle. In some situations it might be desirable to keep the trajectories of a dynamical system with transient chaos far from the attractor and close to this set but its nonattractive nature, the complex dynamics associated with it and eventually the presence of noise may difficult this task. Assume, as an extra difficulty, that our action on the system is bounded and smaller than the action of noise. In such a situation this might seem impossible. However, we will show that in a variety of one dimensional maps this is possible indeed. The control strategy is based on the existence of a set, the safe set, with interesting properties that are due to the same conditions that imply the existence of a chaotic saddle in the system. An example of application of our control technique with the logistic map and some numerical simulations confirming our results are also presented in this work.

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1. Introduction

Some dynamical systems are not chaotic but they present a nonattractive invariant set where the dynamics is chaotic. A manifestation of the existence of that set, usually referred to as chaotic saddle, is the observation of chaotic transients: short periods of time in which the dynamics of a trajectory is chaotic, before it settles to an attractor [2]. Transient chaos is nearly as ubiquitous as chaos itself, and in different contexts [3] it might be desirable to keep the system close to the chaotic saddle in order to avoid the attractor.

Different techniques have been proposed in recent years to achieve this goal. A method inspired in the OGY chaos control scheme [5], based on stabilization of the system around one of the unstable periodic orbits that lie in the chaotic saddle, has shown its effectiveness [8]. Other authors have proposed a method based on applying small perturbations to the return map [3] of continuous-time dynamical systems.

The nonattracting nature of the chaotic saddle, and the erratic behavior of the trajectories that pass nearby, is the main difficulty for the control task. If the system is also affected by noise, staying close to the chaotic saddle might be even more difficult. Imagine, as an extra difficulty, that our action on the system is limited to be smaller than the action of noise. Then, it would seem that it is impossible to remain close to the nonattracting chaotic set. However, in a recent paper Aguirre *et al.* [1] showed that this is indeed possible for the simpler dynamical system with a chaotic saddle and escapes to infinity: the slope three tent map.

The aim of this work is to generalize the results obtained in [1] to a more general class of one dimensional maps presenting a chaotic saddle. We are going to show that, as in [1], paradoxically the same geometry giving rise to the existence of a chaotic saddle will help us to design a strategy to keep the trajectory close to the nonattracting chaotic set by using a control smaller than noise.

The structure of the paper is the following. In Section 2 we state the problem in a precise way and we enounce the main result of this work as a theorem. In Section 3 we present this particular set of points that will help us to design our control strategy, the safe set, and we give as a proposition its main properties. Once we have defined this set and its properties, in Section 4 we prove our main result. Finally, in Section 5 we show an example of application of our technique with the well known logistic map and in Section 6 we draw the main conclusions of our work.

2. Problem statement and main result

First we will define in a precise way the class of dynamical systems that we deal with. We consider one dimensional maps $x_{n+1} = f(x_n)$ where $f : \mathbb{R} \to \mathbb{R}$ is a map that satisfies the following conditions.

(i) There is an interval $I = [a, b]$ such that $I \subset f(I)$. The interval I can be divided in three subintervals $A_1 = [a, x_-]$, $A_0 = (x_-, x_+)$, and $A_2 = [x_+, b]$ such that $f(A_1) = f(A_2) = I$ and $f(A_0) \notin I$.

(ii) The map f is continuous and differentiable in $A_1 \cup A_2$ and for all $x_0 \in A_1 \cup A_2$, $|f'(x_0)| > 1$.

(iii) For all $x_0 \notin I$, $|f^n(x_0)| \to \infty$ as $n \to \infty$.

From conditions (i)–(iii) it can be proved (see [6]) that there is a nonattractive Cantor-like set $\Lambda \subset A_1 \cup A_2$ where the dynamics is topologically equivalent to a shift on two symbols, that is, there is a chaotic saddle. We must point out that condition (ii) is not a necessary condition for the existence of a chaotic saddle. However it makes both the proof of the existence of Λ (see [6]) and the calculations needed in this paper much easier. Note that the slope three tent map $x_{n+1} =$

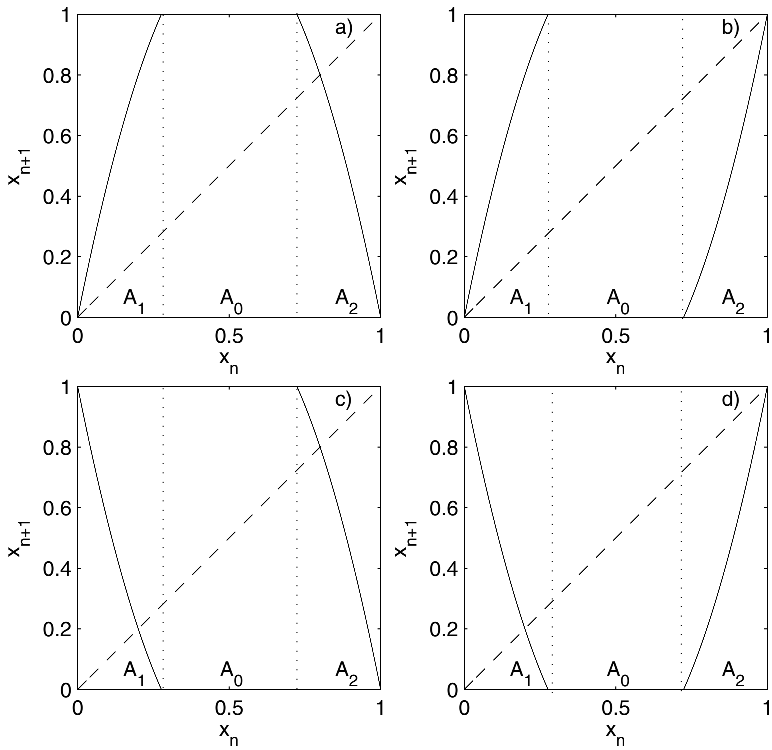

FIGURE 1. Four possible configurations of a map $x_{n+1} = f(x_n)$ satisfiying conditions (i)–(iii). Note that each point in I has just one preimage in A_1 and other in A_2

$3(1 - |x_n|) - 1$ studied in [1] satisfies these conditions. For this particular case, Λ is like the classical middle-third Cantor set constructed using as the starting segment the $[-1, 1]$ interval.

On the other hand, condition (iii) does also imply that for all $x_0 \notin \Lambda$ $|f^k(x_0)| \to \infty$ for $k \to \infty$. If we would have established that all trajectories starting out of I settle to any other type of attractor out of I the existence of a chaotic saddle Λ in $A_1 \cup A_2$ could be established in the same manner. However, we have opted to fix condition (iii) both for simplicity and to make a certain analogy with some chaotic scattering problems, a context in which control of transient chaos is important. For this kind of problems it is well-known [9] that all trajectories except those starting either on a zero-measure invariant set or in its stable manifold diverge from the scattering region to infinity. The same thing applies to the system that we are dealing with: in absence of control only the trajectories starting in Λ will not diverge to infinity under iterations of f.

Once we have defined the type of dynamical system that we will deal with in this letter, we can define in a more precise way the type of situation that we want to control. When controlling a certain dynamical system, specially a physical system, there are two main ingredients that must be considered: first, the deterministic component of its dynamics, and second the eventual presence of a random deviation from the expected deterministic dynamics (the noise). We consider here systems where the deterministic part of the dynamics is modeled by a map $f : \mathbb{R} \to \mathbb{R}$ that satisfies conditions (i)–(iii), so we are considering a system that presents transient chaos. Thus, starting from a point x_n the dynamics of the system takes it to

$$x' = f(x_n)\,.$$

Now we introduce in our model an additive perturbation playing the role of noise, u_n, that deviates the trajectory from its deterministic path, taking it to

$$x'' = x' + u_n = f(x_n) + u_n\,.$$

We assume that u_n is a random number such that $|u_n| \leq u_0$, where $u_0 > 0$.

As we said before, in the system considered nearly all the trajectories (except those lying in Λ) will diverge to infinity in absence of noise. With noise, it is clear that all the trajectories will diverge to infinity. Our objective here is to avoid such divergence to infinity. To do this, we can apply a small perturbation r_n each iteration to control the system's dynamics. Thus, the final state of the system is the result of the action of the deterministic dynamics, modeled by f, of noise, modeled by u_n, and of the small control applied to the system, modeled by r_n. The control r_n is also bounded by a positive constant r_0, so $|r_n| < r_0$ for all n. Thus, each time step, the evolution of the system is given by

$$x_{n+1} = x'' + r_n = f(x_n) + u_n + r_n\,. \tag{2.1}$$

Our aim here is to show that, contrary to what intuition may say, there is a way to keep the trajectories in $A_1 \cup A_2$ (or "close" to the chaotic saddle Λ) even if $r_0 < u_0$. Or, speaking in physical terms, if the control is smaller than noise. To do this, the only thing that we need is to fix the initial condition x_0 accurately. After this, by applying a wisely chosen perturbation r_n each time step, with $|r_n| < r_0 < u_0$ trajectories can be kept bounded *ad infinitum*. This is the main result of this work and it can be stated as follows:

Theorem 2.1 (Main result). *For a dynamical system like the one given by eq.* (2.1)*, where f satisfies conditions* (i)–(iii)*, for all $u_0 > 0$ there is a $0 < r_0 < u_0$ such that $x_n \in A_1 \cup A_2$ for all n.*

This theorem was proved in the particular case of $f(x)$ being the slope three tent map in [1]. Here we prove this theorem for a wider class of one-dimensional maps. As in [1], the key element for this control strategy is the existence of a set with very interesting properties: the safe set. In the next section we will define this set and we will show that its properties can be derived from the same conditions (i)–(iii) that implied the existence of a chaotic saddle. Thus, the main idea of this work is that, contrarily to what intuition might say, the existence of a chaotic saddle does not add an extra degree of difficulty but, instead, it can be of great help for the control task.

3. The safe set and its structure

In this section we define the safe set and explain and prove its main properties. We first define the following maps:

Definition 3.1. *Let f be a map satisfying conditions* (i)–(iii). *Then $F_i \equiv f^{-1}(x) \cap A_i$ for $i = 1, 2$.*

These two maps have a first essential property that will allow us to define the safe sets:

Proposition 3.2. *The map $F_i : I \to A_i$ is a one-to-one map for $i = 1, 2$.*

Proof. From conditions (i) and (ii) it is clear that f is invertible both in A_1 and in A_2, so $F_i : I \to A_i$ is one to one. In other words, any point in I has just one preimage in A_1 and just one preimage in A_2. This can be clearly observed in the examples shown in Fig. 1. □

Thus, given a point $z \in I$, its preimage in A_1 is $F_1(z)$, and its preimage in A_2 is $F_2(z)$. We can now define the safe sets of order k.

Definition 3.3. *Let x_1^0 be the middle point of A_0, that will be called the safe point of order 0. Thus, we can define inductively the safe points of order k, $\{x_i^k\}_{i=1}^{2^k}$ as:*

$$\left\{x_i^k\right\}_{i=1}^{2^k} \equiv f^{-k}\left(x_1^0\right) \cap I = \cup_{i=1}^{2^{k-1}} \cup_{j=1}^{2} F_j\left(x_i^{k-1}\right) \tag{3.1}$$

The set of safe points of order k is called the safe set of order k. The sub index $i \in \{1...2^k\}$ of x_i^k is assigned in such a way that $i < j \leftrightarrow x_i^k < x_j^k$.

From the definition given above, it might seem paradoxical to call these sets the "safe sets" of order k, as long as it is clear from eq. (3.1) that all the elements of this set fall out of I after $k+1$ iterations, after which they will diverge to infinity. However, we will show now that the properties of this set justify this denomination.

A first main property of this set, that can be easily deduced from this definition, is the following: given a point z that belongs to the safe set of order k, then $f(z)$ will belong to a safe set of order $k-1$. This simple property, together with the two following ones, that will be presented as a proposition, are the properties that make these sets of points play a key role for our control strategy.

Proposition 3.4. *Consider the safe sets of order k of a map f satisfying conditions* (i)–(iii). *Then, for all $k \geq 0$ and for all i, $1 \leq i \leq 2^k$:*

– *The safe points of order k and the safe points of order $k+1$ satisfy:*

$$x_{2i-1}^{k+1} < x_i^k < x_{2i}^{k+1}. \tag{3.2}$$

– *Consider the maximum and minimum distance between a safe point of order k and the two adjacent safe points of order $k+1$:*

$$\delta^k_{max} = \max_i \left\{ |x^k_i - x^{k+1}_{2i-1}|, |x^k_i - x^{k+1}_{2i}| \right\} \tag{3.3}$$

$$\delta^k_{min} = \min_i \left\{ |x^k_i - x^{k+1}_{2i-1}|, |x^k_i - x^{k+1}_{2i}| \right\} , \tag{3.4}$$

then

$$\lim_{k\to\infty} \delta^k_{max} = \lim_{k\to\infty} \delta^k_{min} = 0 \,. \tag{3.5}$$

Proof. First we must remember that in the definition of f we assumed that the interval A_1 is to the left of the interval A_2. Thus, for those x^{k+1}_i with $i = 1, ..., 2^k$ and certain j that will depend on i, $x^{k+1}_i = F_i(x^k_j)$. Analogously, for those x^{k+1}_i with $i = 2^k + 1, ..., 2^{k+1}$ and certain j' that will depend on i, $x^{k+1}_i = F_i(x^k_{j'})$. On the other hand we proved that both F_1 and F_2 are monotonous in I. The type of monotonicity will depend on whether f is an increasing or a decreasing function in A_1 and in A_2. In this proof we will assume that given two points $z_1, z_2 \in I$ such that $z_1 < z_2$, then $F_1(z_1) < F_1(z_2)$ and $F_2(z_1) > F_2(z_2)$. This is the case of a map as the one shown in Fig. 1 (a). For the remaining configurations of the map shown in Fig. 1, the proof of this proposition is analogous.

The key observation now is that the only relation between the safe points of order $k+1$ and those of order k that holds with our assumptions is $x^{k+1}_i = F_1(x^k_i)$ for $i = 1, ..., 2^k$ and $x^{k+1}_i = F_2(x^k_{2^{k+1}+1-i})$ for $i = 2^k + 1, ..., 2^{k+1}$. Considering these relations, the proof of this proposition is easy.

We first prove eq. (3.2) inductively. The $k = 0$ case is simple as long as A_1 is to the left of A_2 and A_0 is between these intervals. Thus, $x^1_1 = F_1(x^0_1) < x^0_1 < F_2(x^0_1) = x^1_2$. Assuming that the eq. (3.2) is true for k, we will show that it is true for $k+1$. All we need is to apply F_1 and F_2 to this equation. Equation 3.2 and our assumption on F_1 implies that $F_1(x^{k+1}_{2i-1}) < F_1(x^k_i) < F_1(x^{k+1}_{2i})$ for $i = 1, ..., 2^k$ so, considering the relation given between the safe points of order k and those of order $k+1$, this means that $x^{k+2}_{2i-1} < x^{k+1}_i < x^{k+2}_{2i}$ for $i = 1, ..., 2^k$.

Analogously, to complete the proof of eq. (3.2) we apply F_2 to eq. (3.2) and we have that $F_2(x^{k+1}_{2i-1}) > F_2(x^k_i) > F_2(x^{k+1}_{2i})$ for $i = 1, ..., 2^k$. Considering our observation, this is equivalent to $x^{k+2}_{2^{k+2}-2i+2} > x^{k+1}_{2^{k+1}+1-i} > x^{k+2}_{2^{k+2}+1-2i}$ and, by making the change of index $j = 2^{k+1}+1-i$, it is equivalent to $x^{k+2}_{2j-1} < x^{k+1}_j < x^{k+2}_{2j}$ with $j = 2^k + 1, ...2^{k+1}$. This completes the proof of eq. (3.2).

The proof of eq. (3.5) is also quite simple. We assumed that for all $x_0 \in A_1 \cup A_2$, $|f'(x_0)| > 1$. Thus there are two positive constants $L_{max} > 1, L_{min} > 1$ such that $L_{min} \leq |f'(x_0)| \leq L_{max}$. Then:

$$\begin{aligned}\delta^k_{max} &= \max_i \left\{ |x^k_i - x^{k+1}_{2i-1}|, |x^k_i - x^{k+1}_{2i} \right\} \\ &= \max \left\{ |F_n(x^{k-1}_j) - F_n(x^k_{2j-1})|, |F_n(x^{k-1}_j) - F_n(x^k_{2j})| \right\}\end{aligned}$$

for certain j and for certain $n = 1, 2$. Thus, using the mean value theorem and the bound of the derivative given above:

$$\begin{aligned}\delta^k_{max} &= \max\left\{|F_n(x^{k-1}_j) - F_n(x^k_{2j-1})|, |F_n(x^{k-1}_j) - F_n(x^k_{2j})|\right\} \\ &\leq \frac{1}{L_{min}} \max\left\{|x^{k-1}_j - x^k_{2j-1}|, |x^{k-1}_j - x^k_{2j}|\right\} \leq \frac{\delta^{k-1}_{max}}{L_{min}},\end{aligned}$$

so $\delta^k_{max} \leq \dfrac{\delta^0_{max}}{(L_{min})^k}$ and eq. (3.5) follows. □

Remark 3.5. According to equation 3.2, a safe point of order k has two adjacent safe points of order $k+1$ that are closer to it than any other safe point of order k. Thus, a trajectory lying in a safe point of order $k+1$ is mapped to a point that has a safe point of order $k+1$ to its left and another one to its right. This property is probably the most important one of the safe sets, and it will play a key role in our control strategy.

Once that we have given the key properties of the safe sets, we can now explain our control strategy, which completes the proof of Theorem 2.1.

4. Proof of the main result

Considering the properties given above, we can now give a demonstration of our main result.

Proof of the main result. The only thing that we have to do to control the system with $r_0 < u_0$ is to put the initial condition on a safe point of an accurately chosen order. To find it, we first have to chose k in such a way that $u_0 > \delta^k_{max}$ which, by eq. (3.5) is always possible if k is sufficiently big.

Considering this, we just have to put the initial condition on a safe point of order $k+1$. After this, f maps this point to a safe point of order k, say x^k_i. Then noise acts, and there are two possibilities, according to eq. (3.2):

– That $x^k_i + u_n$ is to the left of x^{k+1}_{2i-1} or to the right of x^{k+1}_{2i}. In this case, considering that the minimum distance between a safe point of order k and the two adjacent safe points of order $k+1$ is δ^k_{min}, a correction r_n such that $|r_n| \leq u_0 - \delta^k_{min}$ will make $x^k_i + u_n + r_n$ lie on a safe point of order $k+1$.
– That $x^k_i + u_n$ is between x^k_i and x^{k+1}_{2i-1} or between x^k_i and x^{k+1}_{2i}. In this case, considering that the maximum distance between a safe point of order k and the two adjacent safe points of order $k+1$ is δ^k_{max}, a correction r_n such that $|r_n| \leq \delta^k_{max}$ will make $x^k_i + u_n + r_n$ lie on a safe point of order $k+1$.

Thus, even if the perturbations r_n are bounded by $r_0 = \max\{u_0 - \delta^k_{min}, \delta^k_{max}\} < u_0$, trajectories starting on a safe point of order $k+1$ can always be placed on a safe point of order $k+1$. This procedure can be repeated forever, which completes our proof. □

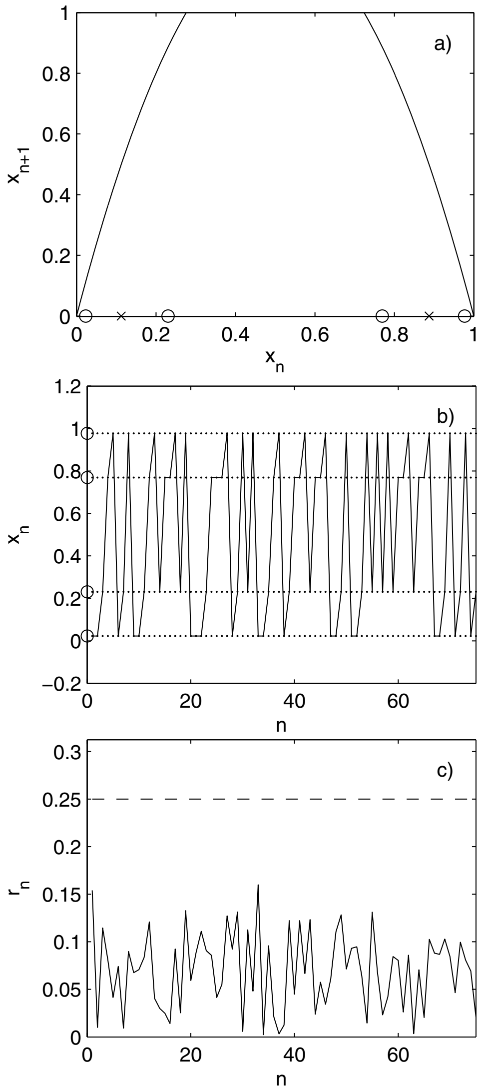

FIGURE 2. The safe points of order 2 ('o') and of order 1 ('×') plotted in the x_n axis together with the curve of the logistic map $x_{n+1} = 5x_n(1 - x_n)$. A trajectory controlled with $u_0 = 0.25$, which is always kept in the safe points of order 2 (marked by 'o' in the x_n axis) (b). The correction applied each iteration, which is always smaller than the maximum perturbation applied $u_0 = 0.25$, marked with a dashed line (c)

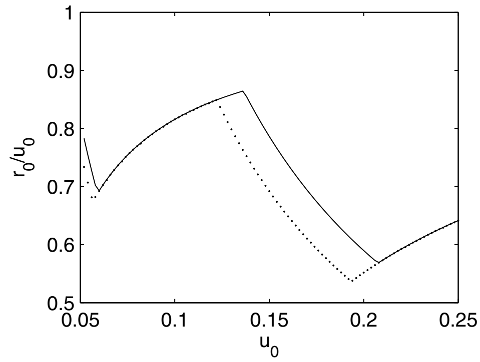

FIGURE 3. The ratio r_0/u_0 obtained for different values of u_0 numerically ('··') and from the analytical expressions ('—'). Note that this ratio is always smaller than 1

Remark 4.1. This theorem does not say a word about which is the optimal k that allows to minimize the ratio r_0/u_0. It is a simple exercise to show that the optimal ratios are bounded by the following quantities:

– The ratio $\dfrac{r_0}{u_0} \leq \dfrac{u_0 - \delta^k_{min}}{u_0}$ if $u_0 \in (\delta^k_{max} + \delta^k_{min}, \delta^{k-1}_{max} + \delta^k_{min}]$.

– The ratio $\dfrac{r_0}{u_0} \leq \dfrac{\delta^k_{max}}{u_0}$ if $u_0 \in (\delta^k_{max} + \delta^{k+1}_{min}, \delta^k_{max} + \delta^k_{min}]$.

Thus, we have shown that in a variety of dynamical systems the same geometrical conditions giving rise to transient chaos have allowed us to define a set, the safe set, with some very interesting properties which, on the other hand, allow to keep the trajectories in the vicinity of the chaotic saddle even if control is smaller than noise. In next section we are going to give an example of application of our control technique using the well-known logistic map.

5. An example of application: Control of transient chaos for the logistic map

In this section we are going to explore our technique in a simple situation, using the well known logistic map $x_{n+1} = \mu x_n(1 - x_n)$. Although it is well known that for $\mu > 4$ this map presents a chaotic saddle [4], which is formed after a boundary crisis, in [6] it is proved that this map satisfies conditions (i)–(iii) just for $\mu \geq 2 + \sqrt{5}$. In the numerical simulations carried out here we will focus on the $\mu = 5$ case.

For this map, $x_- = \dfrac{1}{2} - \dfrac{\sqrt{\mu^2 - 4\mu}}{2\mu}$ and $x_+ = \dfrac{1}{2} + \dfrac{\sqrt{\mu^2 - 4\mu}}{2\mu}$ and thus $x^1_0 = \dfrac{1}{2}$.

As an example, assume first that we perturb the system with a random perturbation that is bounded by $u_0 = 0.25$. We must first find a k such that $u_0 \geq \delta_{max}^k$. We observe numerically that with $k = 1$, this condition is fulfilled. The safe points of order 2 and those of order 1 are shown in Fig. 2 (a), and we can appreciate how they present the expected structure: each safe point of order 1 has two adjacent safe points of order 2.

In Fig. 2 (b) we can observe a controlled trajectory. As we said, the idea is to adjust r_n in such a way that the resulting $x_{n+1} = f(x_n) + u_n + r_n$ lies always on a safe point of order two. The trajectory is kept bounded in 75 iterations and it could be bounded forever. Note that, in absence of perturbations (even of noise), considering that the initial condition lies on a safe point of order two, after three iterations the trajectory would lie out of $[0, 1]$, and then go to infinity. In Fig. 2 (c) we also show the value of the correction applied each iteration, showing that the main result obtained in this paper is observed in this example, as expected. The correction applied in these iterations is always smaller than $u_0 = 0.25$. In fact, we observe that $\max_n (|r_n|)/u_0 \approx 0.15/0.25 = 0.6$, so with a control that is approximately 60 % of the noise the trajectories are kept bounded.

Finally, in Fig. 3 we have shown the bounds of the optimal ratios r_0/u_0 that allow to keep the trajectories bounded ad infinitum, obtained analytically from the expressions given in Remark 4.1, and their numerical estimations, which were obtained by computing the maximum $|r_n|$ necessary to control a trajectory of 10000 time steps. Note that these ratios are always smaller than one, but their value depend on the value of u_0.

6. Conclusions

In this paper we have shown a way to control transient chaos in one dynamical systems using a very particular set of points: the safe sets. A main advantage of this type of control is that, by accurately choosing the initial condition, we can stabilize the system applying perturbations even smaller than the perturbation on the dynamics induced by the presence of noise.

This is due to the very interesting properties of these sets, which themselves can be derived from the same mathematical conditions from which the existence of the chaotic saddle in the dynamical system can be inferred. These conditions are intimately related with the typical "stretching and folding" processes associated with chaotic dynamics and transient chaos. It is well known that this type of process is also present in higher dimensional dynamical systems, like in the paradigmatic Smale horseshoe map [6] and we have recently proved [7] that safe sets also arise in this kind of structures, which are themselves present in a variety of situations. All this makes us think that considering the global geometrical properties of a dynamical system can be useful from a control point of view, not only to control transient chaos but also to control other dynamical situations that involve this type of "stretching and folding" of the phase space.

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