Nonlinear Dynamics
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Partial Control of the Bouncing Ball Map

Samuel Zambrano and Miguel A.F. Sanjuán

Departmento de Física, Universidad Rey Juan Carlos, Tulipán s/n, 28933 Madrid, Spain.

Abstract:

Control of dynamical systems has received a growing interest in last years. Most of the existing control techniques allow to stabilize the system considered in a desired periodic orbit. However, in some situations, we might just need to keep the system's trajectories inside a region from which they will typically escape, without making it follow a prescribed trajectory. We call this type of control *partial control*. Environmental noise might be an obstacle to achieve this goal in practical applications. However, our aim here is to show, using as an example the bouncing ball map, that there is an advantageous partial control strategy that allows to keep trajectories bounded inside a region of phase space where the system acts like a horseshoe map, from which nearly all trajectories diverge (except a zero-measure set). We show here that the particular geometrical action of the map allows to do this even if the control applied is smaller than the noise amplitude. A numerical exploration of the application of this technique to this paradigmatic example is also performed.

1 Introduction

Since Ott, Grebogi and Yorke published their seminal paper [1] on control of chaotic systems, this branch of nonlinear dynamics has attracted growing interest for its potential applicability in different fields of science. The main goal of most of these works is either to find a way to lead the trajectory to any of the unstable periodic orbits that are embedded inside the chaotic attractor by applying small perturbations to the system, or to destroy the chaotic attractor by applying some small harmonic perturbations to the system (see [2,3]).

Although chaotic attractors are widespread, there are other types of complex dynamical behaviour that are also very common in nature, and for which it might also be desirable to achieve some type of control. That is the case of *transient chaos*, a situation where there is a nonattractive set in phase space where the dynamics is chaotic, (a *chaotic saddle*), from which typically all trajectories diverge. Different techniques [4–7] have been proposed with this objective. This type of control is also known as chaos preservation [8] or chaos maintenance [9], and it has been shown to be useful in many different contexts. On the other hand, another widespread phenomena in nonlinear dynamics are *interior crisis* and crisis-induced intermittency [10] by which, if one of the system's parameters is varied, the chaotic attractor is suddenly expanded. After a crisis, the trajectories alternate periods of time in the region of the phase space where the pre-crisis attractor lied with excursions out of it. Those excursions might be undesirable, so that some schemes have also been proposed [11–13] that allow to keep the trajectories close to the region where the pre-crisis attractor lied.

In most of the works mentioned in the former paragraph the aim of the control is not to make the system follow a prescribed trajectory in the phase space. Instead, their aim is to keep the trajectories in a bounded region from which they would typically escape in absence of control, and where the dynamics can be very complicated. This type of control might be what is needed in many contexts and, in order to differentiate it from other more restrictive control methods we call it *partial control* of a chaotic system. It is also quite evident that in practical applications in physics, environmental noise is an obstacle in order to achieve this goal. Our aim in this paper is to give a novel example of an advantageous partial control technique that allows to partially control a wide variety of dynamical systems [14]. More precisely, this technique allows to partially control a variety of dynamical systems that act like a horseshoe map on a region of phase space. Our technique allows to keep trajectories bounded in that region even if the control applied is smaller than the noise amplitude. In this paper, we describe the main features of this technique by using as a novel example of application the well-known bouncing ball map [15].

The structure of this paper is the following: in Section 2 the control problem is stated precisely and the main ideas of our partial control technique are outlined, introducing the key concept of safe set. After this, in Section 3 we present the map that will be used in this paper, the bouncing ball map, and we describe how safe sets can be found in this example. In Section 4 we outline our partial control strategy. After this, in Section 5 we make some numerical analysis of our control technique and finally Section 6 is devoted to the conclusions.

2 The partial control strategy

In this Section we state the partial control problem in a general way. We consider that the unperturbed dynamics of the m-dimensional system considered is given by the one-to-one map $\mathbf{p}_{n+1} = \mathbf{f}(\mathbf{p}_n)$, that can also be a Poincaré map of a $m+1$-dimensional flow in an appropriate surface of section. We assume now that there is a region in phase space Q from which nearly the trajectories (except a zero-measure set) escape under iterations of the map, and where the dynamics might be complex due to the presence of a nonattractive chaotic set (i.e., a chaotic saddle).

As in most physical applications, trajectories might be deviated due to the action of the environmental noise, so the dynamics of the system can be modelled by the equation $\mathbf{p}_{n+1} = \mathbf{f}(\mathbf{p}_n) + \mathbf{u}_n$, where $\mathbf{u}_n$ is a bounded random perturbation, $||\mathbf{u}_n|| \leq u_0$, that plays the role of *noise*.

The aim of partial control is to keep the trajectories inside the region of phase space Q from which trajectories would typically diverge. The only thing that we can do in order to achieve this goal is to apply an accurate *control* $\mathbf{r}_n$ each iteration, that we assume also bounded by a positive constant $||\mathbf{r}_n|| \leq r_0$, in such a way that the global dynamics of our system is given by the equations

$$\begin{cases} \mathbf{q}_{n+1} = \mathbf{f}(\mathbf{p}_n) + \mathbf{u}_n \\ \mathbf{p}_{n+1} = \mathbf{q}_{n+1} + \mathbf{r}_n, \end{cases} \tag{1}$$

(so the control $\mathbf{r}_n$ depends on $\mathbf{p}_n$ and $\mathbf{u}_n$).

The strategy that we use to partially control this type of systems inside Q is the following. First, we are going to locate inside the region Q certain safe sets where trajectories can be steered into with the adequate control $\mathbf{r}_n$ each iteration (although it is not important to know exactly where will they lie). The key point here is that for a wide variety of dynamical systems it is possible to find safe sets with a geometrical structure that allow to do this even if the amplitude of applied control is smaller than the noise amplitude. In other words, it is possible even if $r_0 < u_0$.

In the remaining part of this paper, we are going to make use of the well-known bouncing ball map to show that if a dynamical system acts like a horseshoe map on certain region Q of the phase space, it is possible to find these safe sets in Q. Thus, for these systems it is possible to keep trajectories bounded in Q, from which they would typically escape, even if the control applied is smaller than the noise amplitude.

3 Safe sets for the bouncing ball map

The map that we deal with is the bouncing ball map [15], which is defined as:

$$(\phi_{n+1}, v_{n+1}) = (\phi_n + v_n, \alpha v_n + \beta \cos(\phi_n + v_n)). \tag{2}$$

This map will play the role of the map f in Eq. 3. This map displays very different types of dynamics depending on the values of parameters α and β. However, we are interested here in the values $\alpha = 1$ and $\beta = 12.6$. For these parameters values, it can be proved rigorously that for certain parallelogram Q of the phase space, the map f acts like a horseshoe map. This particular geometrical action is shown in Fig. 1.

As a consequence of this stretching and folding, nearly all the trajectories escape from Q under iterations of the map. If we add noise to the system, then all trajectories eventually escape from Q. Our aim here is to avoid those escapes, and to show that this type of partial control is possible even if we use control that is smaller than the noise amplitude. To do that, we need the safe sets that we already mentioned. Where can they be found?

The safe sets for horseshoe maps can be obtained inductively as follows. Let the safe set S^0 be (somehow paradoxically) the segment in Q that escapes from it under one iteration of f, a segment that goes from the left side to the right side of Q. Now, note that, due to the geometrical action of f^{-1}, for any curve γ from the left to the right side of Q, $\mathrm{f}^{-1}(\gamma) \cap Q$ consists on 2 new curves. Then, the safe set $S^1 = \mathrm{f}^{-1}(S^0) \cap Q$ will consist on two disjoint curves going from the left side to the right side of Q. Thus, $S^2 = \mathrm{f}^{-1}(S^1) \cap Q$ will consist on 4 curves from left to right in Q. Proceeding analogously, the safe set S^k can be generated inductively using the following formula:

$$S^k = \mathrm{f}^{-1}(S^{k-1}) \cap Q. \tag{3}$$

The safe sets S^0, S^1 and S^2 can bee seen in Fig. 1(b). From this figure, and considering also the geometrical action of the bouncing ball map, we can see that these three properties, that are those that make them useful for our control purposes, hold:

(i) The set S^k consists of 2^k curves going from the left side to the right side of Q.

(ii) Each curve of S^k is surrounded by two curves of S^{k+1}. In other words, each curve of S^k has two curves of S^{k+1} that are closer to it than any other curve of S^k.

(iii) We call the maximum and the minimum distance between any curve of S^{k-1} and the two curves of S^k that surround it Δ_k and δ_k respectively. Then,

$$\lim_{k\to\infty} \Delta^k = \lim_{k\to\infty} \delta^k = 0 \tag{4}$$

Now that those safe sets are known, we can show why trajectories can be kept inside Q even if $r_0 < u_0$.

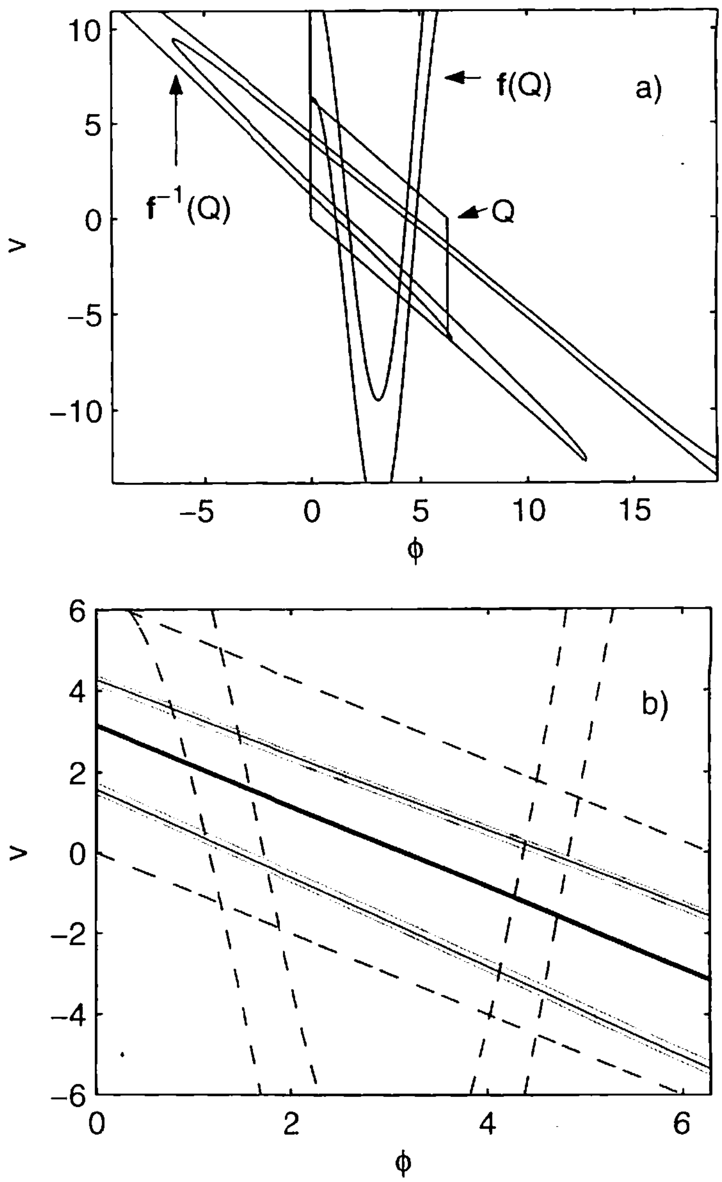

Figure 1: Action of the bouncing ball map on the parallelogram Q (a) and the resulting safe sets S^0 (thick black), S^1 (black) and S^2 (grey) (b), whose geometrical structure will allow to partially control the system inside Q even if $r_0 < u_0$.

4 Partial control strategy for the bouncing ball map

The strategy that allows to keep the trajectory inside the square Q is the following. For simplicity we assume here that u_0 is smaller than the minimum distance between $\mathrm{f}(Q)$ and the left and right sides of Q. Given u_0, find the set S^k such that $\Delta^k < u_0$, which is always possible by Eq. 4. Then, put the initial condition $\mathbf{p}$ in any point on S^k. The action of the map will take the trajectory to $\mathrm{f}(\mathrm{p})$, that by definition will lie in one of the 2^{k-1} curves of S^{k-1}. After this, the noise acts. But, as we can see in Fig. 2, the fact that any curve of S^{k-1} is surrounded by two adjacent curves of S^k allows to use a correction $||\mathrm{r}|| < u_0$

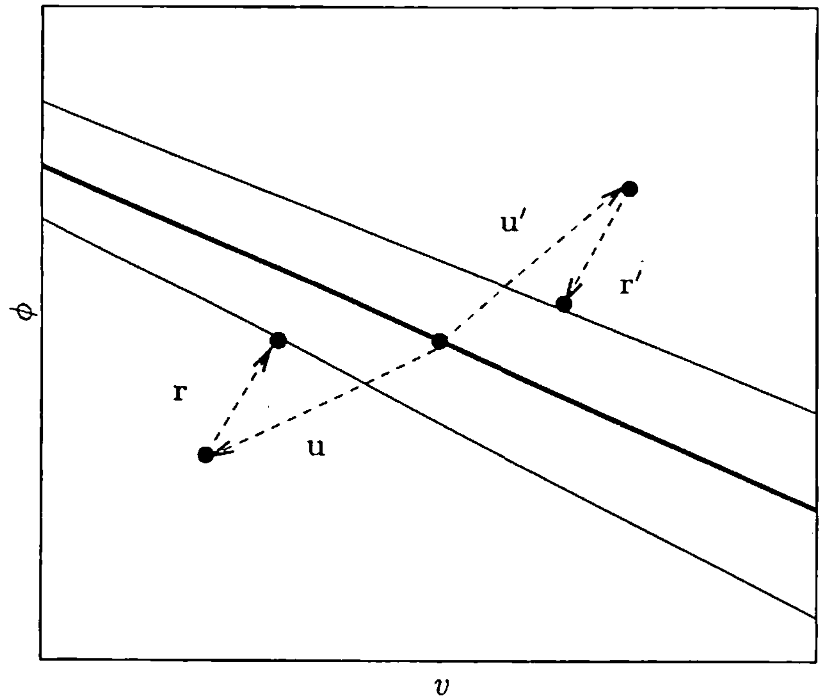

Figure 2: Illustration of the partial control strategy. The trajectory that lied in S^k (thin black line) is mapped into a point on S^{k-1} (thick black line). However, noise (here u or u′) deviates it. But the fact that any curve of S^{k-1} is surrounded by two curves of S^k allows to achieve this goal even if the correction applied (here r or r′) is smaller than the noise amplitude.

to put the resulting point $f(p) + u + r$ back on a point of S^k, and this can be repeated forever. Note that this implies that we can find a value of the control r_0 such that trajectories can be kept inside Q even if $r_0 < u_0$.

The power of our technique can be evaluated by estimating the value of the ratio r_0/u_0 needed to control the system. Although an explicit calculation of this ratio is difficult in most cases, by having a sufficiently accurate estimation of the values of Δ^k and δ^k it is possible to make an analytical estimate of the maximum r_0/u_0 needed to partially control the system for a given value of u_0. In fact, it can be proved that

- The ratio needed to control the system if $u_0 \in (\Delta^k + \delta^{k+1}, \Delta^k + \delta^k]$ is bounded by $\frac{r_0}{u_0} \leq \frac{\Delta^k}{u_0}$.
- The ratio needed to control the system if $u_0 \in (\Delta^k+\delta^k, \Delta^{k-1}+\delta^k]$ is bounded by $\frac{r_0}{u_0} \leq \frac{u_0 - \delta^k}{u_0}$.

n the next section we show that these expressions allow to have quite good approximations of the ratio ıeeded to control the system for each value of u_0.

; Numerical example

ıs a final example of application, we can show how our control technique works for a given value of $_0$, for example $u_0 = 0.35$. For this value of the noise amplitude the adequate safe set turns out to be

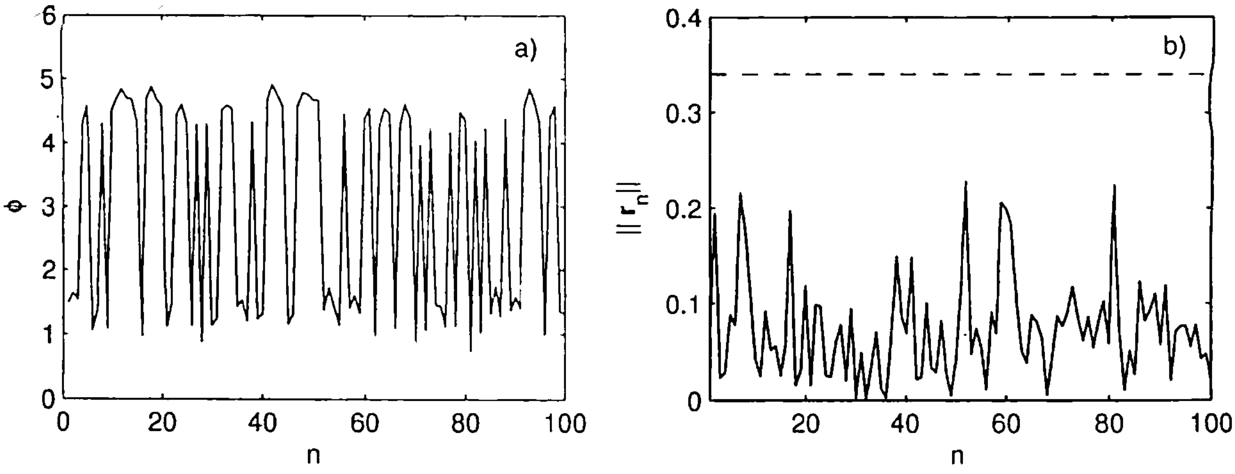

Figure 3: A controlled series of the bouncing ball map for $u_0 = 0.32$ (a) and the control applied each iteration (b), that is obviously smaller than u_0 (marked with a dashed line).

S^2, that is shown in Fig. 1. An example of a controlled time series of the variable ϕ is shown in Fig. 3 (a), and it is easy to see that trajectories are kept inside the parallelogram. On the other hand, in Fig. 3 (b) we can see that, in order to steer the trajectories into S^2, a control such that $||\mathbf{r}_n|| \leq 0.2$ is enough. Thus, with a control that is approximately 57 % of the noise amplitude, trajectories can be kept bounded forever.

We have also made some simulations in order to see the validity of the approximations of the ratios r_0/u_0 needed to keep trajectories bounded done in Section 4. We have picked 100 different time series of the system for different values of the noise amplitude u_0 and we have estimated the minimum value of r_0 needed to keep trajectories bounded. The results of these calculations can be seen in Fig. 4. We can notice there that there is a good agreement between the analytical and the numerical estimates. In order to perform the analytical estimate of the ratio r_0/u_0 we need to know the values of Δ^k and δ^k. This is difficult to do analytically, so in this case we have estimated them numerically. In any case, the shape of the curves that might be expected from our analysis is in good agreement with the purely numerical results.

6 Practical issues and conclusions

Some remarks are necessary. In order to apply our technique, we need to locate the safe sets. This implies a previous knowledge of the region of the phase space where the horseshoe acts like a horseshoe map. We expect this to be feasible by doing a time series analysis of the system, although we do not know clear references on how this should be exactly done. Location of homoclinic points and homoclinic intersections can be a good starting point. However, it is important to notice that just an approximate knowledge of the position of the safe sets is needed to adequately partially control the considered system. This can be easily proved by using some of the ideas sketched here. From our point of view this problem is equivalent to an inaccuracy of the applied control. However, even in presence of moderate inaccuracies the ratio r_0/u_0 needed to partially control the system can be kept smaller than one.

In summary, we used the bouncing ball map to illustrate that for a wide and important class of two

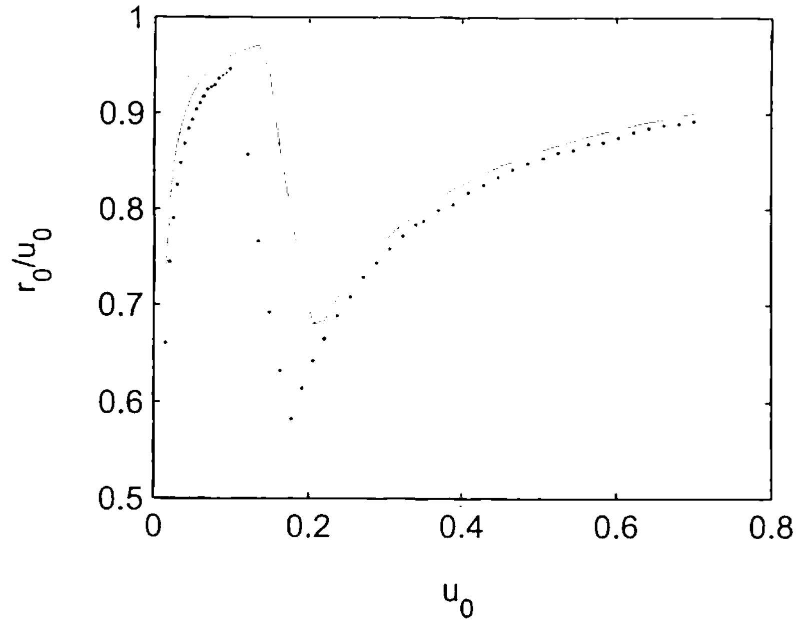

Figure 4: Ratio r_0/u_0 needed to partially control the system for different values of u_0, computed numerically (black dots) using 100 different time series of 10000 time steps and its analytical estimate (grey line). Note that the estimates are quite similar to the real values and that the computed values of r_0/u_0 are always smaller than one, as claimed.

dimensional dynamical systems, there is a strategy by which the trajectories can be kept in a region of phase space where a horseshoe-like map acts even with a control that is smaller than the noise. Paradoxically, this is due to the same general geometrical conditions that make nearly all the trajectories diverge from that region, that also imply the existence of certain sets, the safe sets, with a very interesting structure. We have shown a numerical example of application of our method. In our opinion, the wide presence of this kind of structures in dynamical systems allow to apply this technique to a wide variety. A more general lesson that could be extracted from our work is that the particular geometry that is usually related with certain types of complex dynamics may be useful for control purposes.

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