Chapter 14

CHAOS STABILIZATION IN THE THREE BODY PROBLEM

Arsen R. Dzhanoev[1,*], Alexander Loskutov[1],
James E. Howard[2] and Miguel A. F. Sanjuán[3]

[1] *Physics Faculty, Moscow State University, 119992 Moscow, Russia*

[2] *Laboratory for Atmospheric and Space Physics and Center for Integrated Plasma Studies University of Colorado, Boulder CO 80309 USA*

[3] *Departamento de Fisica, Universidad Rey Juan Carlos, 28933 Mostoles, Madrid, Spain*

[*] *janoev@polly.phys.msu.ru*

A new type of orbit in the three-body problem is constructed. It is analytically shown that along with the well known chaotic and regular orbits in the three-body problem there also exists a qualitatively different type of orbit which we call "stabilized". The stabilized orbits are a result of additional orbiting bodies that are placed close to the triangular Lagrange points. The results are well confirmed by numerical orbit calculations.

1. Introduction

The three-body problem appears in practically all fields of contemporary physics from studies on microscopic systems to macroscopic ones: quantum mechanics [1], ionic oscillations [2], protein-folding [3], planetary systems formation [4] etc. The problem considers three particles of mass m_i with positions $\mathbf{r}_i$ which are each moving under an attractive force from all the other bodies where particle index $i = 1, 2, 3$. The system is characterized by a set of differential equations

$$m_i \ddot{\mathbf{r}}_i = \sum_{j=1, i\neq j}^{3} \frac{\gamma m_i m_j (\mathbf{r}_j - \mathbf{r}_i)}{|\mathbf{r}_i - \mathbf{r}_j|^3}, \tag{1}$$

where γ is the gravitational constant that, by appropriately choosing dimensions, could be ignored by setting it equal to unity. These equations define the phase flow in an 18 dimensional phase space. Exploiting the symmetry afforded by (1) leads to a 12 dimensional phase space with 10 integrals of motion. These are the only known integrals. Henri Poincaré suggested that highly complex behavior could occur in the the three-body problem. It is reasonable, if we cannot find a general solution, to examine special solutions and particular features (see for example [5]). A natural starting place is the restricted three-body problem. In the restricted three-body problem, m_3 is taken to be small enough so that it does not influence the motion of m_1 and m_2 (called primaries), which are assumed to be in circular orbits about their center of mass. For the restricted three-body problem it was analytically verified that the complex behavior is due to the existence of transverse heteroclinic points. A well-known example of the chaoticity of the restricted three-body problem is the Sitnikov problem.

2. The Sitnikov Problem

The Sitnikov problem consists of two equal masses M (primaries) moving in circular or elliptic orbits about their common center of mass and a third test mass μ moving along the straight line passing through the center of mass normal to the orbital plane of the primaries (see Fig. 1).

The circular problem was considered first by McMillan in 1913 [6]. He found the exact solution of the equations of motion when the eccentricity of the primaries $e = 0$ and showed that it can be expressed in terms of elliptic integrals. Detailed discussion on this case has been done by Stumpff [7]. This problem became important when Sitnikov [8] in 1960 investigated the elliptic case of $e > 0$ and proved the possibility of the existence of oscillatory motions which were earlier predicted by Chazy in 1922–32. Alekseev [9] in 1968–69 proved that in the Sitnikov problem all of the possible combinations of final motions in the sense of Chazy are realized. Later in 1973 the alternative proof of the Alexeev results was done by Moser [10]. Since then the Sitnikov problem has attracted the attention of many other authors. Here we mention some of them. An interesting work in a qualitative way was carried out by Llibre and Simó [11] in 1980 and later by C.Marchal [12] in 1990. Hagel [13] derived an approximate solution of the differential equation of motion for particle μ by using a Hamiltonian in action angle variables. An explicit numerical study of the great variety of possible structures in phase space for the Sitnikov problem has been done by Dvorak [14]. Using

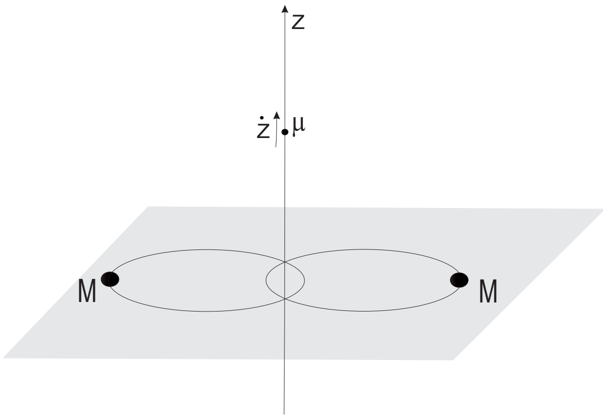

Fig. 1. Geometry of the Sitnikov problem.

Melnikov's method H.Dankowicz and Ph.Holmes [15] were able to show the existence of transverse homoclinic orbits. They proved that for any but a finite number of values of the eccentricity e the system is non-integrable, chaotic.

The main objective of this Chapter is to show through analytical and numerical methods the existence of stabilized orbits in this special restricted three-body problem and consequently, in the general three-body problem. The equation of motion can be written, in scaled coordinates and time as

$$\ddot{z} + \frac{z}{[\rho(t)^2 + z^2]^{3/2}} = 0, \tag{2}$$

where z denotes the position of the particle μ along the z-axis and $\rho(t) = 1 + e\cos(t) + O(e^2)$ is the distance of one primary body from the center of mass. Here we see that the system (2) depends only on the eccentricity, e, which we shall assume to be small.

We first consider the circular Sitnikov problem i.e. when $e = 0$, for which

$$\begin{aligned} H &= \frac{1}{2}v^2 - \frac{1}{\sqrt{1+z^2}} \\ v &= \dot{z}. \end{aligned} \tag{3}$$

The level curves $H = h$, where $h \in [-2, +\infty)$, partition the phase space (v, z) into qualitatively different types of orbits as shown in Fig. 2. We are interested in solutions that correspond to the level curves $H = 0$, namely two parabolic orbits that separate elliptic and hyperbolic orbits and can be

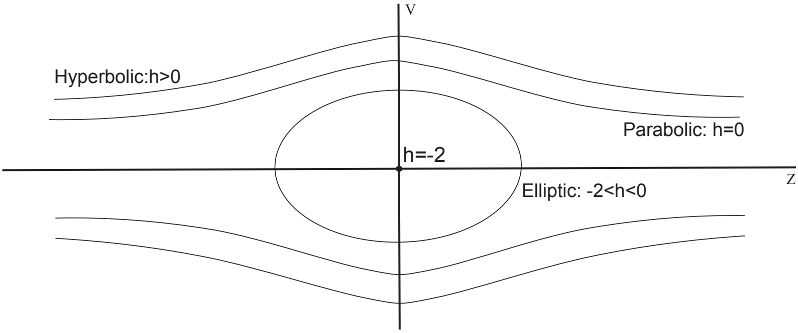

Fig. 2. Phase portrait for the unperturbed Sitnikov problem.

considered as a separatrix between these two classes of behavior. Further from Eq. (2) it is easy to show that there are just two fixed points: (0,0) — the center at the origin and $(\pm\infty, 0)$. Making use of the McGehee transformation [15] the fixed points at $(\pm\infty, 0)$ correspond to hyperbolic saddle points. Then taking into account that parabolic orbits act as connections or heteroclinic orbits between these two fixed points one may conclude that the stable and unstable manifolds of saddles correspond to the parabolic orbits.

To make clear how this problem is related to heteroclinic orbits, let us employ the non-canonical transformation [15]:

$$z = \tan u, \qquad v = \dot{z}, \qquad u \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], v \in R. \tag{4}$$

Then the Hamiltonian for the equation (2) in the new variables (u,v) has the form:

$$H(u,v) = \frac{1}{2}v^2 - \frac{1}{\rho(t)^2 + \tan^2 u} = H_0(u,v) + eH_1(u,v,t,e), \tag{5}$$

where $H_0(u,v) = \dfrac{1}{2}v^2 - \cos u$. One can see that when $e = 0$ the form of the Hamiltonian that obtained after the non-canonical transformation exhibits the pendulum character of motion.

Based on this connection between the dynamics of the nonlinear pendulum and the Sitnikov problem one can show that if $e \in (0,1)$ then for all but a possibly finite number of values of e in any bounded region, the system (2) is chaotic [15]. In this work we consider only small values of e. Hence due to the KAM-theory [16], since our system has 3/2 degrees of freedom the invariant tori bound the phase space and chaotic motion is finite and takes place in a small vicinity of a separatrix layer.

3. Stabilization of Chaotic Behavior in the Vicinity of a Separatrix

As mentioned in the introduction, our analysis is directed to the stabilization of this chaotic behavior in the elliptic Sitnikov problem. In general, this problem is related to the stabilization and control of unstable and chaotic behavior of dynamical systems by external forces. Since there are situations for which chaotic behavior might be undesirable, different methods have been developed in the past years to suppress or control chaos. The idea that chaos may be suppressed goes back to the publications [17, 18] where it has been proposed to perturb periodically the system parameters. The method of controlling chaos has been introduced in the paper [19] (the history of this question see in review [20]). A comprehensive study of chaotic systems with external controls was done in [21, 22]. Further we will give a brief review of these results. In this section we apply the Melnikov method, which gives a criterion of the chaos appearance, to the analysis of the system behavior under external perturbations. The idea is that such an approach can give us an analytical expression of the perturbations which leads to the chaos suppression phenomenon.

We explain the idea by using a general two-dimensional dynamical system subjected to a time-periodic external perturbation, and consequently possessing a three dimensional phase space.

3.1. *Melnikov function*

It is well known that in Hamiltonian systems, separatrices can split. In this case stable and unstable manifolds of a hyperbolic point do not coincide, but intersect each other in an infinite number of homoclinical points (usually the motion in the $(n+1)$–dimensional phase space $(x_1, \cdots, x_n, t)$ is considered in the projection onto a n–dimensional hypersurface $t = \text{const}$ (Poincaré section)). The presence of such points gives us a criterion for the observation of chaos. This criterion can conveniently be obtained by the *Melnikov function* (MF), which "measures" (in the first order of a small perturbation parameter) the distance between stable and unstable manifolds.

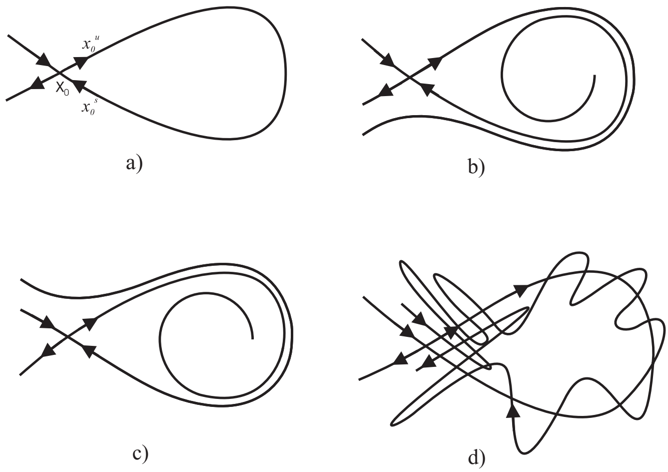

Fig. 3. Poincaré section $t = \text{const} \pmod{T}$ of the system (Eq. 6) for $\varepsilon = 0$ (Fig. 3(a)) and $\varepsilon \neq 0$ (Figs. 3(b) to 3(d)).

Melnikov analysis is based on the paper [23]. First, we consider a two-dimensional dynamical system under the action of a periodical perturbation with the property of having a unique saddle point:

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}) + \varepsilon \mathbf{f}_1(\mathbf{x}, t), \tag{6}$$

Let furthermore $\mathbf{x}_0$ be the separatrix of the unperturbed system $\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x})$. Then the MF at any given time t_0 is defined as follows:

$$D(t_0) = -\int_{-\infty}^{+\infty} \mathbf{f}_0 \wedge \mathbf{f}_1 \bigg|_{\mathbf{x}=\mathbf{x}_0(t-t_0)} dt,$$

where the integral is taken along the unperturbed separatrix $\mathbf{x}_0(t - t_0)$ and the integrand is $\mathbf{f}_0 \wedge \mathbf{f}_1 = f_{0x} f_{1y} - f_{0y} f_{1x}$.

In general, in dissipative systems one can observe three possibilities for the MF: either $D(t_0) < 0$ (Fig. 3(b)), $D(t_0) > 0$ (Fig. 3(c)) for any t_0 or $D(t_0)$ changes its sign for some t_0 (Fig. 3(d)). Only in the last case chaotic dynamics arises. Thus, the MF determines the character of the motion near the separatrix. Note that the Melnikov method has a perturbative (to first order) character, thus, its application is allowed only for trajectories which are sufficiently close to the unperturbed separatrix.

3.2. *Function of stabilization*

The Melnikov method has been applied in a lot of typical physical situations (see Refs. [24–29]) in which homoclinic bifurcations occur. Here we consider an application of the Melnikov method to the analysis of the chaos suppression phenomenon in systems with separatrix loops. Such an approach allows us to find an analytical expression of the perturbations for which the Melnikov distance $D(t_0)$ does not change sign (see also [30]) suppressing the chaotic behavior and stabilizing the orbits of the system.

We consider the problem of stabilization of chaotic behavior in systems with separatrix contours that can be described by Eq. (6)

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}) + \varepsilon \mathbf{f}_1(\mathbf{x}, t),$$

where $\mathbf{f}_0(\mathbf{x}) = (f_{01}(\mathbf{x}), f_{02}(\mathbf{x}))$, $\mathbf{f}_1(\mathbf{x}, t) = (f_{11}(\mathbf{x}, t), f_{21}(\mathbf{x}, t))$. For this equation the Melnikov distance $D(t_0)$ is given by $D(t_0) = -\int\limits_{-\infty}^{\infty} \mathbf{f}_0 \wedge \mathbf{f}_1 dt \equiv I[g(t_0)]$. Let us assume that $D(t_0)$ changes its sign. To suppress chaos we should get *a function of stabilization* $\mathbf{f}^*(\omega, t)$ that leads us to a situation where separatrices do not intersect:

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}) + \varepsilon \left[\mathbf{f}_1(\mathbf{x}, t) + \mathbf{f}^*(\omega, t)\right], \tag{7}$$

where $\mathbf{f}^*(\omega, t) = (f_1^*(\omega, t), f_2^*(\omega, t))$. Suppose $D(t_0) \in [s_1, s_2]$ and $s_1 < 0 < s_2$. After the stabilizing perturbation $\mathbf{f}^*(\omega, t)$ is applied we have two cases: $D^*(t_0) > s_2$ or $D^*(t_0) < s_1$, where $D^*(t_0)$ is the Melnikov distance for system (7). We consider the first case (analysis for the second one is similar). Then

$$I[g(t_0)] + I[g^*(\omega, t_0)] > s_2, \tag{8}$$

where $I[g^*(\omega, t_0)] = -\int\limits_{-\infty}^{+\infty} \mathbf{f}_0 \wedge \mathbf{f}^* dt$. Expression (8) is true for all left hand side values of inequality that is greater than s_2. It is derived that $I[g(t_0)] + I[g^*(\omega, t_0)] = s_2 + \chi = const$, where $\chi, s_2 \in \mathbb{R}^+$. Therefore $I[g^*(\omega, t_0)] = \text{const} - I[g(t_0)]$. On the other hand, $I[g^*(\omega, t_0))] = -\int\limits_{-\infty}^{\infty} \mathbf{f}_0 \wedge \mathbf{f}^* dt$. We choose $\mathbf{f}^*(\omega, t)$ from the class of functions that are absolutely integrable on an infinite interval such that they can be represented in Fourier integral form. Then $\mathbf{f}^*(\omega, t) = \text{Re}\{\hat{A}(t) e^{-i\omega t}\}$. Here we suppose that $\hat{A}(t) = (A(t), A(t))$ i.e., assume that the regularizing perturbations applied to both components of Eq. (7) are identical. Therefore $-\int_{-\infty}^{\infty} \mathbf{f}_0 \wedge \left\{\hat{A}(t)e^{-i\omega t}\right\}dt = \text{const} - I[g(t_0)]$. The inverse Fourier transform yields: $\mathbf{f}_0 \wedge \hat{A}(t) = \int_{-\infty}^{\infty} (I\left[g\left(t_0\right)\right] - \text{const})\, e^{i\omega t} d\omega$. Hence,

$$A(t) = \frac{1}{f_{01}(x) - f_{02}(x)} \int_{-\infty}^{\infty} (I\left[g\left(t_0\right)\right] - \text{const})\, e^{i\omega t} d\omega. \tag{9}$$

Here $A(t)$ can be interpreted as the amplitude of the "stabilizing" perturbation. Thus, for system (6) the external stabilizing perturbation has the form:

$$f^*\left(\omega, t\right) = \text{Re}\left[\frac{e^{-i\omega t}}{f_{01}(x) - f_{02}(x)} \int_{-\infty}^{\infty} (I\left[g\left(t_0\right)\right] - \text{const})\, e^{i\omega t} d\omega\right]. \tag{10}$$

Here it is significant to note that in the conservative case: const=0.

Let us now consider the stabilization problem for systems of the type

$$\begin{aligned} \dot{x} &= P(x, y), \\ \dot{y} &= Q(x, y) + \varepsilon[f(\omega, t) + \alpha F(x, y)], \end{aligned} \tag{11}$$

where $f\left(\omega, t\right)$ is a time periodic perturbation, $P(x, y)$, $Q(x, y)$, $F(x, y)$ are some smooth functions and α is the dissipation.

We investigate the case which is typical for applications with a single hyperbolic point at the origin $x = y = 0$ when $P(x, y) = y$. Let $x_0(t)$ be the solution on the separatrix. In the presence of the perturbation the Melnikov distance $D(t_0)$ for the system (11) may be written as

$$D(t_0) = -\int_{-\infty}^{\infty} y_0(t - t_0)\left[f(\omega, t) + \alpha F(x_0, y_0)\right] dt \equiv I[g(\omega, \alpha)], \tag{12}$$

where $y_0\left(t\right) = \dot{x}_0\left(t\right)$. Let us suppose again that the Melnikov function (12) changes sign, i.e., the separatrices intersect. We will find an external regularizing perturbation $\mathbf{f}^*(\omega, t) = \text{Re}\{\hat{A}(t)e^{-i\omega t}\}$ that stabilizes the system dynamics:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= Q(x, y) + \varepsilon[f(\omega, t) + \alpha F(x, y) + f^*\left(\omega, t\right)]. \end{aligned} \tag{13}$$

It is significant to note that since the system (11) depends on parameter α then such stabilization should be made at every fixed value of this parameter and further, instead of $I[g(\omega, \alpha)]$, we will write $I[g(\omega)]$. For (13) we have $f_{01} = y$, $f_{02} = Q(x, y)$ and $\hat{A}(t) = (0, A(t))$. Consequently the value $A(t)$ has a form

$$A(t) = \frac{1}{y_0(t - t_0)} \int_{-\infty}^{\infty} (I[g(\omega)] - \text{const})\, e^{i\omega t} d\omega. \tag{14}$$

So, for (13) the stabilizing function can be represented as

$$f^*(\omega, t) = \text{Re}\left[\frac{e^{-i\omega t}}{y_0(t - t_0)} \int_{-\infty}^{\infty} (I[g(\omega)] - \text{const})\, e^{i\omega t} d\omega\right]. \tag{15}$$

Now, let us find a regularizing perturbation in the case when the Melnikov function $D(t, t_0)$ admits an *additive* shift from its critical values.

Again, we analyze the case when $D^*(t_0) > s_2$ is satisfied. Suppose that α_c corresponds to the critical value of the Melnikov function, $I_c = I[g(\omega, \alpha|_{\alpha=\alpha_c})]$. Then, a subcritical Melnikov distance can be expressed as $I_{out} = I_c - a$, where $a \in \mathbb{R}^+$ is constant. Assuming that the system perturbed by $f^*(\omega, t)$ exhibits regular behavior, we have

$$I' = I_{out} + I[g^*(\omega)] > s_2. \tag{16}$$

Here $I[g^*(\omega)] = -\int_{-\infty}^{+\infty} y_0(t - t_0)\, f^*(\omega, t)\, dt$. On the other hand, it is obvious that we can take any I' a fortiori greater than I_c:

$$I' = I_c + a > s_2. \tag{17}$$

Now, equating the left-hand sides of (16) and (17), we obtain $I[g^*(\omega)] = 2a$. Substituting $f^*(\omega, t) = \text{Re}\{A(t)e^{i\omega t}\}$ into the expression for $I[g^*(\omega)]$, we find $-\int_{-\infty}^{\infty} e^{i\omega t} A(t) y_0(t - t_0) dt = 2a$. The inverse Fourier transform yields $A(t)\, y_0(t - t_0) = -2a \int_{-\infty}^{\infty} e^{-i\omega t} d\omega$. Hence,

$$A(t) = -\frac{2a}{y_0(t - t_0)} \int_{-\infty}^{\infty} e^{-i\omega t} d\omega = -\frac{4\pi a \delta(t)}{y_0(t - t_0)}. \tag{18}$$

Thus, the dynamics of the systems that admit an additive shift from the critical value of the Melnikov function $D(t_0)$ are regularized by the perturbation:

$$f^*(\omega, t) = -\frac{4\pi a\delta(t)}{y_0(t-t_0)}\cos(\omega t), \tag{19}$$

where $\delta(t)$ is a Dirac delta–function defined as follows:

$$\delta(t) = \begin{cases} 0, & t \neq 0, \\ \infty, & t = 0. \end{cases}$$

In the general case, if $f_0 = (f_{01}(x), f_{02}(x))$, then we obviously obtain

$$f^*(\omega, t) = -\frac{4\pi a\delta(t)}{f_{01}(x) - f_{02}(x)}\cos(\omega t). \tag{20}$$

From the physical point of view the dynamics of the chaotic system are stabilized by a series of "kicks". This result could be easily extended on case when the stabilizing function $f^*(\omega, t)$ is Gaussian function. The orbit that was chaotic and became regular under the influence of the external perturbation we call the *stabilized orbit.*

4. Stabilization of Chaotic Behavior in the Extended Sitnikov Problem

In the vicinity of the orbiting primaries there exist five equilibrium points lying in the $z = 0$ plane. The points L_1, L_2, L_3 are unstable and collinear with the primaries, while each of L_4 and L_5 forms an equilateral triangle with the primaries and are stable, depending on the mass of the primaries. Let us now consider two bodies of mass m that are placed in the neighborhood of the stable triangular Lagrange points of the Sitnikov problem (see Fig. 4). Here we treat only the hierarchical case: $\mu \ll m < M$. In the new configuration that constitutes the extended Sitnikov problem a particle of mass μ experiences forces from the primaries and masses m placed close to L_4 and L_5. These forces are perpendicular to the primaries plane and therefore the particle's motion remains on the z axis. Since the bodies of mass m orbit around their common center of mass, their distance ρ' alternates between ρ'_{min} (periastron) and ρ'_{max} (apastron), consequently the forces between these bodies and the particle μ increase in a close encounter to the barycenter and vanish as the bodies move away from z axis. So we can achieve the situation where the influence of bodies that are placed in the

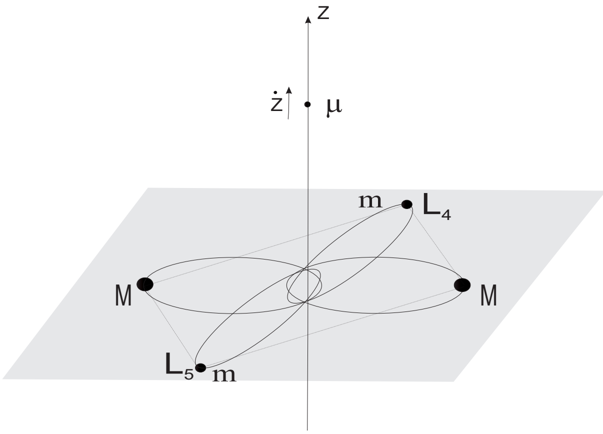

Fig. 4. Geometry of the Extended Sitnikov problem.

vicinity of the triangular Lagrange points of the particle μ can be presented as a series of periodic Gaussian function-like impulses.

Recall that earlier we showed how the elliptical Sitnikov problem deals with heteroclinic orbits that lead to non-integrability due to the existence of transverse heteroclinic points. It was done through the connection between this problem and the pendular character of motion described by (5). Therefore taking into account the new configuration of the restricted three-body problem (Fig. 4) we may say that there is a connection between the extended elliptical Sitnikov problem and the motion of the chaotic nonlinear pendulum with an external impulse-like perturbation. The Hamiltonian of such system changes to

$$H(u,v) = H_0(u,v) + e\left[H_1(u,v,t,e) + H_1^*(u,v,t,e)\right], \tag{21}$$

where $H_1^*(u,v,t,e)$ - the part of Hamiltonian that is responsible for impulsive forces that the particle μ experiences from bodies in the neighborhood of L_4 and L_5.

Now taking into account the result of the previous paragraph we conclude that the forces which the particle experiences from bodies in the neighborhood of L_4 and L_5 act on the chaotic behavior of μ as an external stabilizing perturbation and the system (21) represents the system with the stabilized chaotic behavior that corresponds to the stabilized orbits in the extended Sitnikov problem. As mentioned before, in the circular Sitnikov

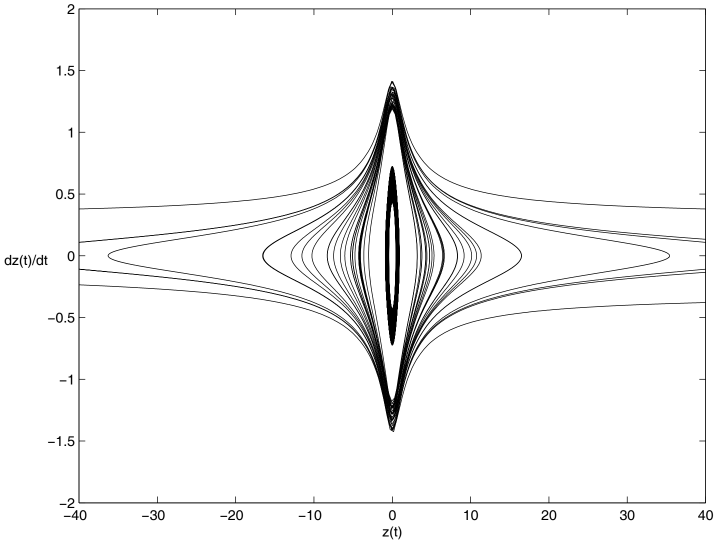

Fig. 5. Phase portrait of the particle motion in Sitnikov problem.

problem when $e = 0$ the phase space is partitioned into invariant curves corresponding to different energies. For $e > 0$ this structure is broken. This is apparent for eccentricity $e = 0.07$ in Fig. 5: just a few invariant curves survive. In this figure we also can see hyperbolic and parabolic orbits that correspond to energy $h \geq 0$. These orbits escape to infinity with positive or zero speed respectively. Now, if we consider the extended Sitnikov problem then one can see (Fig. 6) that all orbits are in a bounded region and there are no escape orbits. So one may infer that stabilized orbits of the pendulum system with Hamiltonian (21) correspond to the stabilized orbits in the extended Sitnikov problem, thus confirming the conclusion that we made before. The extension of the analysis carried out above to the corrections of higher order in ε of Eq. (2) and numerical verification of the obtained results could be found in [31].

In summary, we have performed a study of the existence of a qualitatively different orbit from those previously known in the three-body problem: the stabilized orbits. On the basis of the elliptic Sitnikov problem we constructed a configuration of five bodies which we called the extended Sitnikov problem and showed that in this configuration along with chaotic and regular orbits the stabilized orbits could be realized.

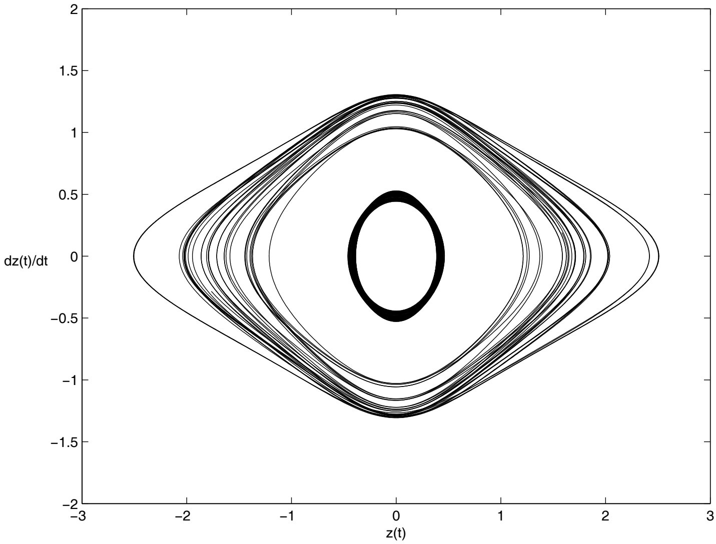

Fig. 6. Phase portrait of the stabilized particle orbits in the extended Sitnikov problem.

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