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STRONG SENSITIVITY OF THE VIBRATIONAL RESONANCE INDUCED BY FRACTAL STRUCTURES

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We consider a nonlinear system perturbed by two harmonic forcings of different frequencies. The slow forcing drives the system into an oscillatory regime while the fast perturbation enhances the effect of the slow periodic drive. The vibrational resonance occurs when this enhancement is optimal, usually when the fast perturbation has an amplitude much higher than the slow periodic forcing. We show that this resonance can also happen when the amplitude of the fast perturbation is far below the amplitude of the slow periodic forcing due to a peculiar condition of the phase space. Moreover, this resonance presents an extreme sensitivity to small variations of the fast perturbation. We explore here this phenomenon that we call *ultrasensitive vibrational resonance.*

Keywords: Vibrational resonance; fractal; delay.

1. Introduction

Vibrational resonance (VR) occurs when the response of a nonlinear system with a low frequency oscillatory signal is optimized by means of a high frequency perturbation [Landa & McClintock, 2000]. The VR has been thoroughly studied analytically, numerically and experimentally in a variety of nonlinear systems [Chizhevsky *et al.*, 2003; Ullner *et al.*, 2003; Deng *et al.*, 2010; Shi *et al.*, 2010; Rajasekar *et al.*, 2012]. Among these studies, the analysis of the effect of delay on the VR has been receiving increasing attention [Yang & Liu, 2010a, 2010b, 2010c, 2011; Jeevarathinam *et al.*, 2011; Daza *et al.*, 2013].

In this context, we have explored the VR in the vicinity of a Hopf bifurcation induced by a delay feedback. Delay differential equations, though very complicated from an analytical point of view, are very easily simulated numerically and display a variety of outstanding phenomena. It is well known that among other interesting effects on the VR, time lags can induce multiresonance responses [Yang & Liu, 2010b; Jeevarathinam *et al.*, 2011]. However, we did not expect to find infinite resonances displaying a fractal pattern, as it has happened. Moreover, this resonance takes place for values of the amplitude of the high frequency perturbation smaller than the amplitude of the low frequency signal. This is a unique feature that previous studies on VR have been missing out.

According to this unusual pattern of the resonance curve, we call the phenomenon *ultrasensitive* *vibrational resonance* (UVR). This resonance is extremely sensitive to slight variations of the amplitude of the high frequency perturbation. Actually, the resonance curves present very sharp and narrow peaks arranged in a fractal pattern in such a way that it might be possible to find two peaks of resonance arbitrarily close. This is far different from the usual continuous bell shaped curve observed in the vibrational resonance where the amplitude of the second forcing spans a large interval of values.

Investigating the origin of this extreme sensitivity, we found that the key underlying property which gives rise to the UVR is the appearance of fractal structures in the phase space. We believe that this kind of structures and the presence of attractors of different amplitude are the basic ingredients of the UVR. Small perturbations in a fractal phase space can lead the system from a small attractor to an attractor of large amplitude, therefore the resonance is induced. To check this hypothesis we analyze a nonlinear system without delay but with a highly fractalized phase space, and we show that this system is also able to display the phenomenon of UVR.

2. Vibrational Resonance

As a starting point, we reproduce the results of [Jeevarathinam *et al.*, 2011], where VR in a Duffing oscillator with a linear delayed feedback has been studied. The Duffing oscillator is a paradigmatic model to search for VR as it was the model chosen in the original article of Landa and McClintock [2000]. Thus, the model for our study can be formulated as follows

$$\begin{aligned}\ddot{x} + \gamma\dot{x} + \alpha x + \beta x^3 + cx(t-\tau) \\ = A\cos\omega t + B\cos\Omega t,\end{aligned} \quad (1)$$

where all the coefficients $\gamma > 0, \alpha < 0, \beta > 0, c < 0, \tau > 0, A > 0, B > 0, \Omega > 0, \omega > 0$, and $\Omega \gg \omega$ are real constants.

This model is the usual Duffing oscillator, with two periodic forcings of different frequencies $\Omega \gg \omega$ and a time-delayed feedback $cx(t-\tau)$.

The aim of the VR is to optimize the response of the system to the frequency ω when the high-frequency perturbation is applied. To quantify this response we need to compute the sine and cosine components of the output signal,

$$C_s = \frac{2}{nT}\int_0^{nT} x(t)\sin\omega t\,\mathrm{d}t \quad (2)$$

$$C_c = \frac{2}{nT}\int_0^{nT} x(t)\cos\omega t\,\mathrm{d}t. \quad (3)$$

Here n is the number of complete oscillations of the low frequency signal and $T = 2\pi/\omega$ is its period. The numerical values of C_s and C_c are related to the Fourier spectrum of the time series of the variable x computed at the frequency ω. Then, the relation between the output and the forcing signals provides an idea of how the low frequency signal is being amplified by the high frequency perturbation. This is commonly defined by means of the response amplitude Q:

$$Q = \frac{\sqrt{C_s^2 + C_c^2}}{A}. \quad (4)$$

The standard procedure to search for the VR consists of computing Q for different amplitudes B of the high frequency perturbation [Landa & McClintock, 2000]. If there is a value of B that maximizes Q, then the VR occurs. This means that there is a particular value of the high frequency periodic perturbation that optimizes the response of the system to the weak low frequency periodic signal. In case of multiresonance the system presents several maxima.

So the search of the VR requires different steps of computation:

- First, we solve the delay differential Eq. (1) using a $(5, 6)$ pair of Runge–Kutta formulas [Thompson & Shampine, 2006]. Delay differential equations require an infinite set of initial conditions called history, which describes the state of the system in a time interval equal to the delay. Here, we choose constant histories set onto a point of equilibrium. This means that for every $t \in [-\tau, 0]$, where τ is the delay, we have

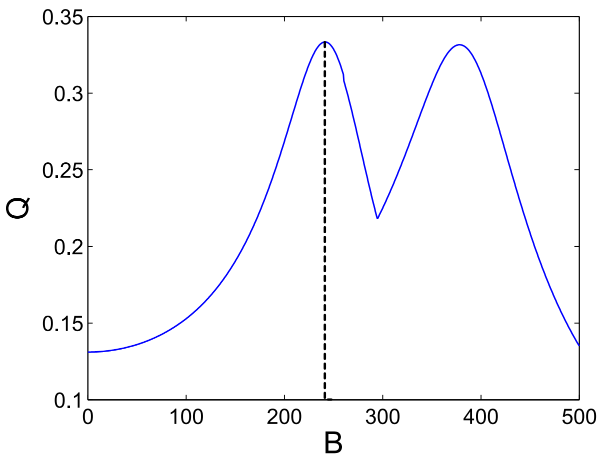
$$\dot{x} = 0, \quad (5)$$

$$x = \pm\sqrt{\frac{\alpha - \gamma}{c}}. \quad (6)$$

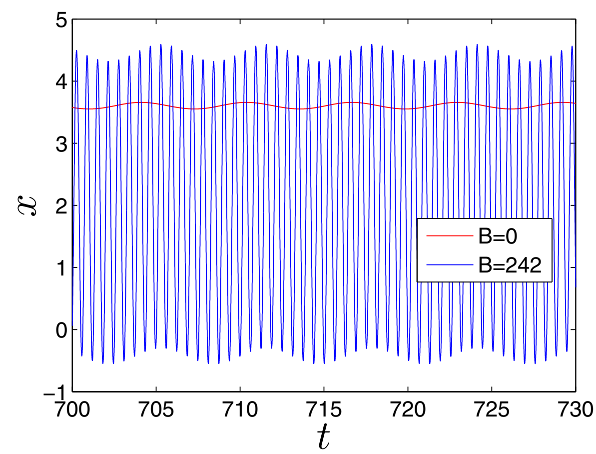
This is completely equivalent to solve the delay differential equations in the absence of external signals with any initial histories, and then apply the two periodic signals after the transient has vanished. Intuitively, a system which remains at equilibrium in time has a clear physical meaning, while other histories are more difficult to interpret. According to this, setting the history at one of the equilibrium points seems the most reasonable choice.

- Once we have the solution of the delay differential equation, we make the Fourier Transform of the time series and calculate the amplitude response Q. The whole process is repeated for a range of different values of the high frequency perturbation intensity B.
- Finally we plot Q versus B. The maxima of this curve, if any, correspond to the VR, that is an optimal match between the low frequency and high frequency signals.

The Duffing oscillator with time-delayed feedback, Eq. (1), can present two resonances corresponding to the two maxima of the Q versus B curve [Fig. 1(a)], for certain parameters [Jeevarathinam *et al.*, 2011]. It is remarkable that in this case, the range of values of the amplitude of the high frequency perturbation B is several orders of magnitude larger than the amplitude of the low frequency signal A. Indeed, the analysis of the theoretical approach of Q includes the assumption $A \ll 1$,



(a)



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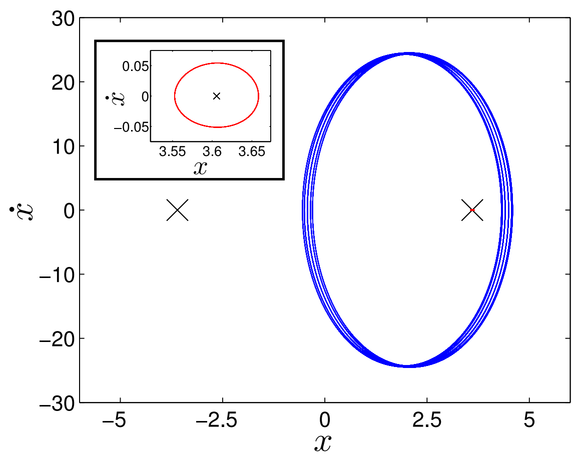


(c)

Fig. 1. (a) Usual VR curve in a time-delayed system, as it appears in [Jeevarathinam *et al.*, 2011]. The amplitude response of the system Q varies smoothly when the amplitude of the high frequency perturbation B is varied and the two maxima correspond to the vibrational resonance. Equation (1) has been solved with the following parameters $\ddot{x} + 0.5\dot{x} - x + 0.1x^3 - 0.3x(t-1) = 0.1\cos t + B\cos 10t$ and histories $x = \dot{x} = 0$ for $t \in [-\tau, 0]$. Notice the wide range of values of the high frequency amplitude B compared to the value of the low frequency amplitude. (b) The time series in red corresponds to $B = 0$ and the time series in blue corresponds to the first maximum of panel (a) marked with a dashed line, which is the usual vibrational resonance. (c) We can see the same time series represented in phase space $(x, \dot{x})$. The apparent thickness of the trajectory is a consequence of the high period regime (actually several lines may appear when we zoom in). The trajectory for the case $B = 0$ is plotted in the inset because it is very small compared to the resonant trajectory. This does not mean that the system presents a great resonance, in the sense of great amplification with a small external action, but this is a consequence of such a large difference between the amplitudes of both periodic signals.

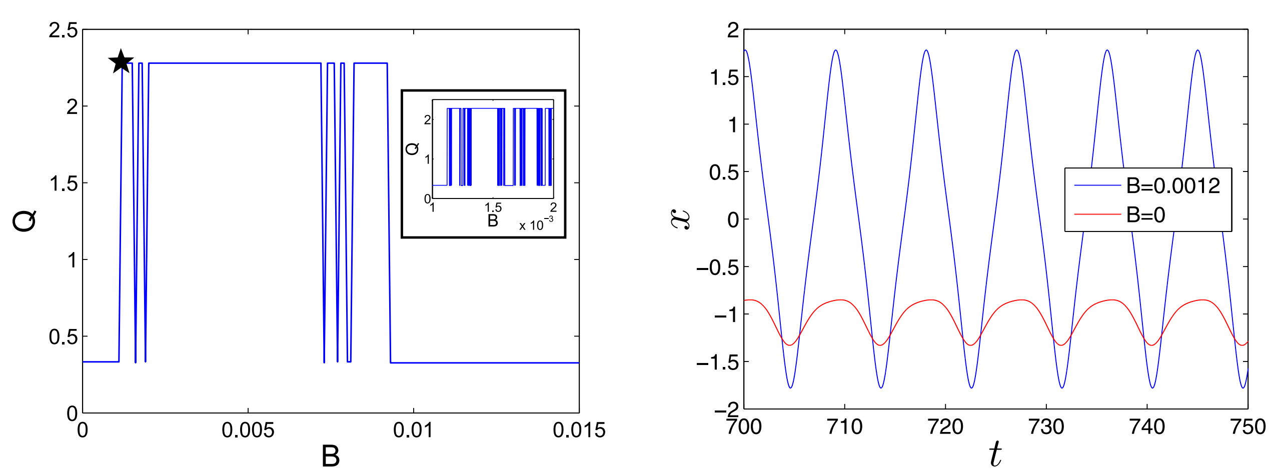
while the value of B that produces the resonance is proportional to Ω, i.e. $B \gg A$.

In these conditions, it is still fair to talk about resonance since the high frequency perturbation enhances the response amplitude Q, but probably one would not expect the cause of the resonance to be much larger than the signal itself. Furthermore, the time series at the resonance resembles the high frequency perturbation acting as an enhancer, and the low frequency signal is completely eclipsed [Figs. 1(b) and 1(c)]. Therefore, it seems desirable to amplify the low frequency signal by means of smaller amplitudes of the secondary high frequency perturbation. As we found out, this is in fact happening for Eq. (1) when the system is on the edge of stability giving rise to the UVR.

3. Ultrasensitive Vibrational Resonance

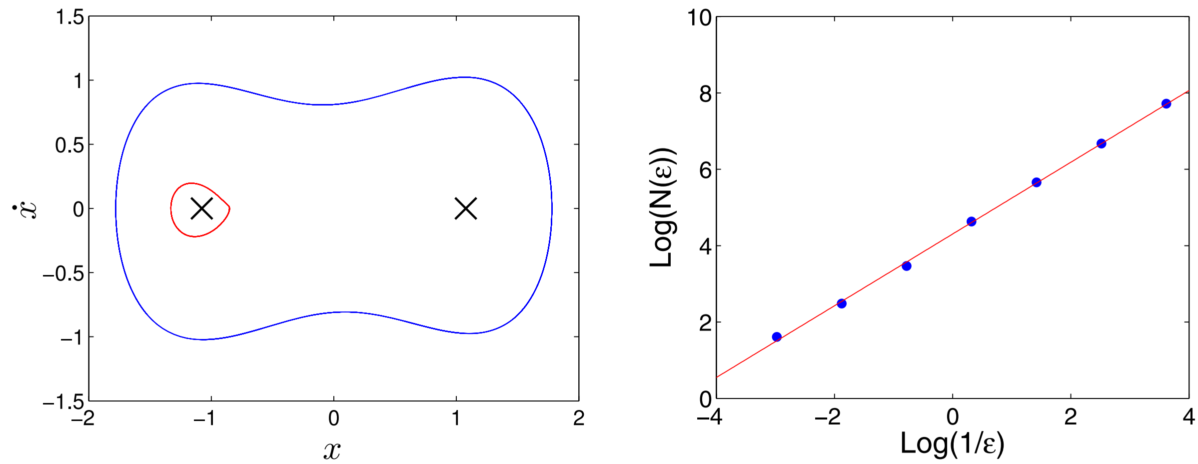
The UVR consists of a series of sharp and narrow peaks of resonance that appear for very small values of the amplitude of high frequency signal, as shown in Fig. 2(a). Moreover, and unlike the common VR presented in the previous section, the UVR occurs for values of high frequency amplitude B which are smaller than the low frequency amplitude A. Furthermore, the final time series are not completely disturbed by the high frequency perturbation as in the VR. However, in the UVR, the resonant time series resemble the low frequency signal but with a much larger amplitude as can be observed in Figs. 2(b) and 2(c). According to this, the UVR fits better the idea of resonance as a big oscillation amplitude driven by a sufficiently small external action.

Another specific feature of this phenomenon is the fractal pattern displayed by the peaks of resonance. This means that when the resolution in B is increased, the amplitude response Q presents more and more maxima. Every peak is actually composed of many peaks and valleys that form a fractal curve, making the resonance extremely sensitive to small changes in the parameters. The height of the peaks of Q is almost constant as it is intimately related to the size of the attractors, which do not vary for such small perturbations. A computation of the box-counting dimension is carried out in order to quantify the fractalization of the resonance curve, leading to a noninteger dimension of $d = 0.94$ [see Fig. 2(d)].



(a) (b)

Fig. 2. (a) Ultrasensitive vibrational resonance curve for equation $\ddot{x} + 0.1\dot{x} - 0.5x + 0.5x^3 - 0.08x(t - 6.3) = 0.174\cos 0.7t + B\cos 3t$. The amplitude response of system Q varies in a sharp manner when the amplitude of the high frequency perturbation B is slightly modified. The inset is a zoom of the first apparent peak, revealing that it is composed of more peaks in a fractal-like structure. The height of Q remains almost constant, as it is very closely related to the amplitude of the attractor, which does not vary appreciably for this short range of B. (b) Time series for $B = 0.0012$, marked with a star in panel (a). Here the resonant series resembles the nonresonant series, but with a larger amplitude. (c) We can see the same time series represented in phase space $(x, \dot{x})$. Notice that we get a strong amplification of the signal, i.e. a high resonance for a very small amplitude of the high frequency perturbation. (d) Computation of the box-counting dimension for the curve of resonance shown in panel (a). The slope of the log–log plot indicates a noninteger box-counting dimension of $d = 0.93965 \pm 0.00016$, which confirms that the curve is fractal.



(c) (d)

Fig. 2. (*Continued*)

The Duffing oscillator with time-delayed feedback of Eq. (1) is able to present the usual VR as shown earlier, but can also display the UVR if the appropriate parameters are chosen. Research in other models of similar characteristics show that when the delay takes values just before the Hopf bifurcation, very small periodic perturbations induce the UVR. Moreover, this phenomenon occurs when the low frequency signal has a frequency similar to the natural oscillations of the system, as it happens in the resonance of linear systems. Delving deeper into the causes, we found that the key element for the appearance of UVR is the fractalization of the phase space that occurs for this set of parameters.

The phenomenon of UVR is better understood by examining the phase space. One procedure to examine the phase space in a delayed system consists in choosing the history as a function with two parameters, and then compute the 2D basin of attraction varying these parameters. Among all the possible functions that can play the role of history for the Duffing oscillator with delay, here we choose the histories as constants values of x and $\dot{x}$ for $t \in [-\tau, 0]$. For every pair of constant histories $(x, \dot{x})$ we integrate the system and plot the basin of attraction, as shown in Fig. 3. This subspace of the infinite phase space of the delayed system, is sufficient to show that fractal structures appear for this particular choice of parameters. Fractal structures associated to transient chaos are an outstanding feature very common in time delay systems [Aguirregabiria & Etxebarria, 1987; Losson *et al.*, 1993; Yin *et al.*, 1995; Taylor & Campbell, 2007]. In this case, we can see in Fig. 3 that the equilibrium point chosen as constant history lies very close to the fractal boundary where three different basins coexist. Although the equilibrium is still stable for these parameters, very small amplitudes of the high frequency perturbation can lead the system to a different basin. In particular, the system can be driven to an attractor of large amplitude oscillations, that is the ultimate cause of the ultrasensitive vibrational resonance.

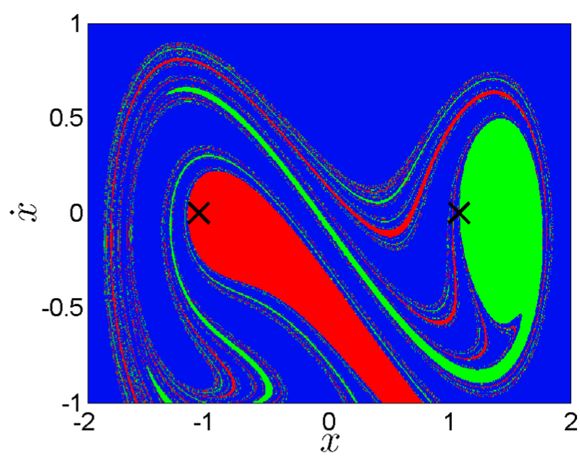


Fig. 3. Basin of attraction for the Duffing oscillator with time-delayed feedback $\ddot{x} + 0.1\dot{x} - 0.5x + 0.5x^3 - 0.08x(t - 6.3) = 0.174 \cos 0.7t$, which corresponds to Eq. (1) with $B = 0$, that is, before the high frequency perturbation is introduced. Histories have been chosen as constants. We can see a fractalization of the projection of the phase space (the actual phase space is infinite-dimensional due to the delay). Perturbations of small amplitude, such as those produced by the high frequency forcing, may drive the system to different attractors.

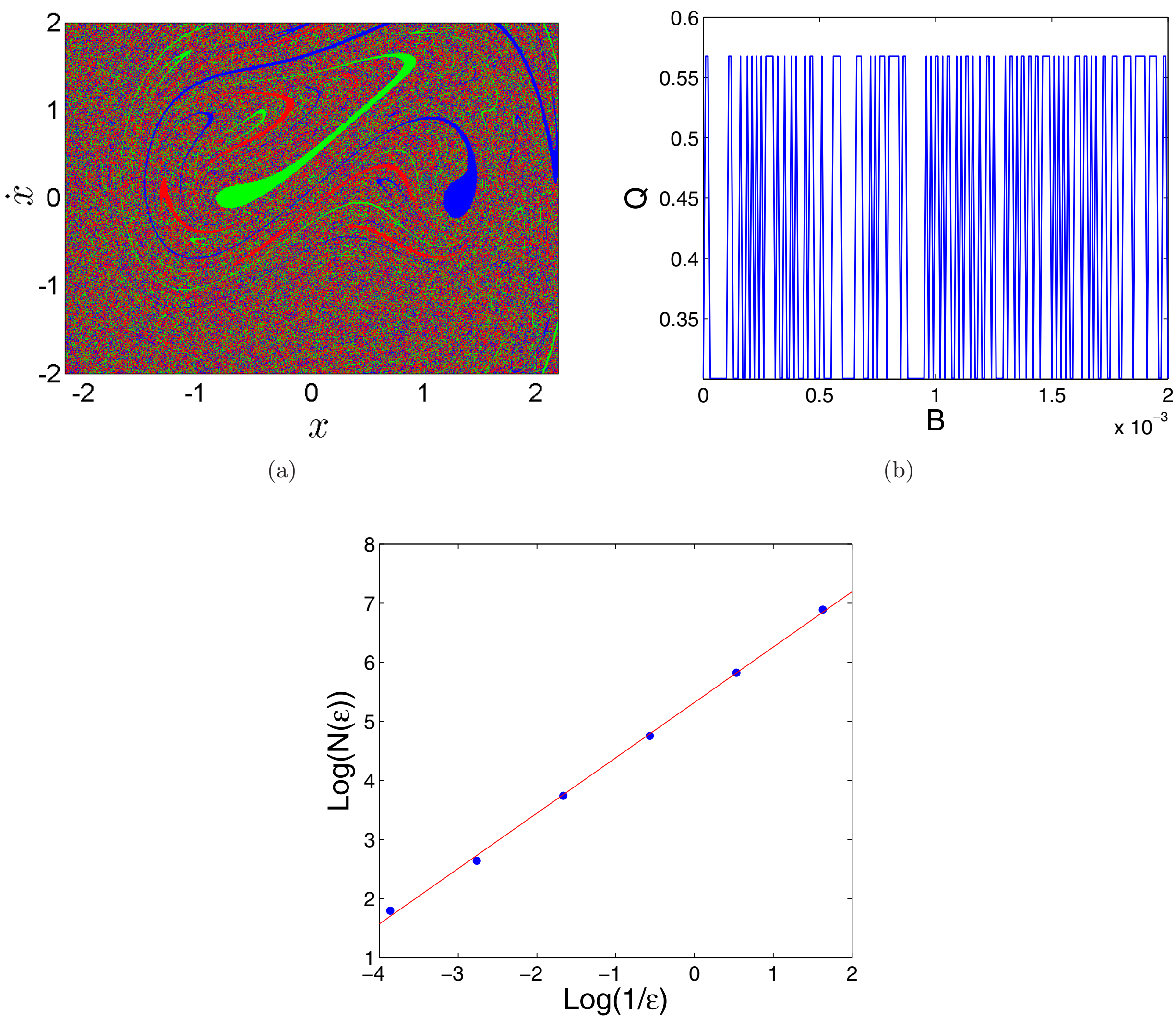
To prove the validity of this interpretation, i.e. actually the effect of the high frequency perturbation is to drive the trajectory to a larger amplitude attractor, we chose another set of parameters for the same model without delay, so we can extend the results to other kind of systems. Here we consider the Duffing oscillator with the following parameters:

$$\ddot{x} + 0.15\dot{x} - x + x^3 = 0.245\cos t. \tag{7}$$

In this system there is no delay at all, but the phase space is highly fractalized [Aguirre & Sanjuán, 2002], as shown in the basin of attraction of Fig. 4(a). In this case the system presents three periodic attractors, two of them of period 1 with small amplitudes and one of period 3 of larger amplitude. Now we introduce the second harmonic perturbation and we have the equation

$$\ddot{x} + 0.15\dot{x} - x + x^3 = 0.245\cos t + B\cos 10t. \tag{8}$$

If we choose the initial conditions to be in a fractal boundary and then compute the response amplitude Q [see Fig. 4(b)], the UVR takes place with the



(a) (b)

(c)

Fig. 4. The Duffing oscillator without delay is able to show UVR for this particular choice of parameters: $\ddot{x}+0.15\dot{x}-x+x^3 = 0.245\cos t$. (a) This figure shows the basin of attraction where we can observe that the phase space is highly fractalized (actually Wada basins [Aguirre & Sanjuán, 2002]). (b) Plot of the ultrasensitive vibrational resonance when we introduce the second harmonic perturbation and then the equation becomes $\ddot{x} + 0.15\dot{x} - x + x^3 = 0.245\cos t + B\cos 10t$. The peaks of resonance follow a fractal-like structure due to the fractalization of the phase space. As in the case with delay of Fig. 2, the height of Q remains almost constant. This is related with the amplitude of the largest attractor of the system. (c) Computation of the box-counting dimension of the curve Q shown in panel (b). The slope of the log–log plot is 0.93737 ± 0.00024, thus the resonance curve has a noninteger dimension.

same characteristics described before. Once again, the high frequency perturbation is able to drive the system to an attractor of large amplitude producing the resonance. This confirms our conviction that the appearance of fractal structures in phase space is the ultimate cause of the UVR. Furthermore, it explains why in both cases, with and without delay, the amplitude response Q takes almost constant values, since the attractor remains unchanged for these small perturbations.

4. Concluding Remarks

A new phenomenon that we call ultrasensitive vibrational resonance is presented in this paper. UVR is a particular case of vibrational resonance with some specific characteristics that make it specially interesting.

In its origin, vibrational resonance was considered as a phenomenon similar to stochastic resonance [Gammaitoni *et al.*, 1998] but with a high frequency perturbation used to amplify the low frequency signal instead of the noise. Probably one would not expect the noise to be larger than the signal, and the same reasoning would be applicable to high frequency perturbation in VR. However, in the previous literature [Yang & Liu, 2010a, 2010b, 2010c, 2011; Jeevarathinam *et al.*, 2011; Daza *et al.*, 2013] the high frequency perturbation typically has a larger amplitude than the low frequency signal. This is not the case in the ultrasensitive vibrational resonance, which can be achieved with very small amplitudes of the high frequency perturbation, even smaller than the amplitude of the low frequency signal.

Besides the small amplitude of the high frequency needed to achieve the resonance, another striking feature of this phenomenon is the fractal pattern of sharp and narrow peaks of resonance. As we zoom on the response amplitude Q, more and more peaks are found as in a fractal curve. We also have computed the box-counting dimension showing that it is not an integer, which confirms its fractal nature.

As an attempt to understand the origin of this high sensitivity of the resonance, we have studied the phase space of the system. We have observed that the Duffing oscillator with a time-delayed feedback presents a phase space with fractal structures that give rise to the phenomenon of UVR. When the initial condition lies on a fractal boundary or very close to it, the high frequency perturbation can drive the trajectory to different attractors. If one of these attractors is of similar frequency to the low frequency signal but with a larger amplitude, then the UVR is possible. This explains the high sensitivity to small variations and also the fractal pattern of the peaks of resonance, which is due to the fractal nature of the phase space. Furthermore, to check this hypothesis, we have studied the same system without delay for a choice of parameters where the basin of attraction is highly fractalized. We have reproduced the same results for the response amplitude Q, proving that the fractal nature of the phase space is at the heart of this phenomenon. This opens the range of systems susceptible to presenting this behavior.

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References

Aguirregabiria, J. M. & Etxebarria, J. R. [1987] "Fractal basin boundaries of a delay-differential equation," *Phys. Lett. A* **122**, 241–244.

Aguirre, J. & Sanjuán, M. A. F. [2002] "Unpredictable behavior in the Duffing oscillator: Wada basins," *Physica D* **171**, 41–51.

Chizhevsky, V. N., Smeu, E. & Giacomelli, G. [2003] "Experimental evidence of vibrational resonance in an optical system," *Phys. Rev. Lett.* **91**, 220602.

Daza, A., Wagemakers, A., Rajasekar, S. & Sanjuán, M. A. F. [2013] "Vibrational resonance in a time-delayed genetic toggle switch," *Commun. Nonlin. Sci. Numer. Simulat.* **18**, 411–416.

Deng, B., Wang, J., Wei, X., Tsang, K. M. & Chan, W. L. [2010] "Vibrational resonance in neuron populations," *Chaos* **20**, 013113.

Gammaitoni, L., Hänggi, P., Jung, P. & Marchesoni, F. [1998] "Stochastic resonance," *Rev. Mod. Phys.* **70**, 223–287.

Jeevarathinam, C., Rajasekar, S. & Sanjuán, M. A. F. [2011] "Theory and numerics of vibrational resonance in Duffing oscillators with time-delay feedback," *Phys. Rev. E* **83**, 066205.

Landa, P. S. & McClintock, P. V. [2000] "Vibrational resonance," *J. Phys. A: Math. Gen.* **33**, L433–L438.

Losson, J., Mackey, M. C. & Longtin, A. [1993] "Solution multistability in first order nonlinear delay differential equations," *Chaos* **3**, 167–176.

Rajasekar, S., Used, J., Wagemakers, A. & Sanjuán, M. A. F. [2012] "Vibrational resonance in biological nonlinear maps," *Commun. Nonlin. Sci. Numer. Simulat.* **17**, 3435–3445.

Shi, J., Huang, C., Dong, T. & Zhang, X. [2010] "High-frequency effects on vibrational resonance in a synthetic gene network," *Phys. Biol.* **7**, 036006.

Taylor, S. R. & Campbell, S. A. [2007] "Approximating chaotic saddles for delay differential equations," *Phys. Rev. E* **75**, 046215.

Thompson, S. & Shampine, L. F. [2006] "A friendly Fortran DDE solver," *Appl. Num. Math.* **56**, 503–516.

Ullner, E., Zaikin, A., García-Ojalvo, J., Bascones, R. & Kurths, J. [2003] "Vibrational resonance and vibrational propagation in excitable systems," *Phys. Lett. A* **312**, 348–354.

Yang, J. H. & Liu, X. B. [2010a] "Controlling vibrational resonance in a multistable system by time delay," *Chaos* **20**, 033124.

Yang, J. H. & Liu, X. B. [2010b] "Delay induces quasi-periodic vibrational resonance," *J. Phys. A: Math. Theor.* **43**, 122001.

Yang, J. H. & Liu, X. B. [2010c] "Controlling vibrational resonance in a delayed multistable system driven by an amplitude-modulated signal," *Phys. Scr.* **82**, 025006.

Yang, J. H. & Liu, X. B. [2011] "Delay-improved signal propagation in globally coupled bistable systems," *Phys. Scr.* **83**, 065008.

Yin, H. W., Dai, J. H. & Zhang, H. J. [1995] "Fractal basin boundaries and transversal heteroclinic intersections in a hybrid optical bistable system," *Phys. Lett. A* **206**, 370–376.

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