CHAPTER 31

FURTHER PROGRESS IN PARTIAL CONTROL OF CHAOTIC SYSTEMS

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The Partial Control technique is a new paradigm in the control of chaotic systems in the presence of noise. This new approach of control allows one to keep trajectories of a dynamical system inside a region from which nearly all the trajectories diverge. Its main advantage is that this goal is achieved even if the corrections applied to the trajectories (control) are smaller than the action of environmental disturbances (noise) on the dynamics. This is a rather counterintuitive result, that is achieved thanks to what we call safe sets. Here, we study the use of the Partial Control technique in one of the most famous chaotic maps, the Hžnon map, and the deep relationship between the safe sets and the sets of points with different escape times, the escape time sets. We also show how it is possible to find certain extended safe sets that can be used instead of the safe sets in the Partial Control technique. We also discuss briefly the development of a new algorithm to find these new safe sets in any dynamical system with a chaotic behaviour.

1. Introduction

Some dynamical systems do not present a chaotic attractor but, instead, they present a nonattractive set in phase space where the dynamics is chaotic: a chaotic saddle. In fact, it is common to find that for a system with a chaotic attractor, by varying one of the system's parameters, a boundary crisis might occur, by which the chaotic set becomes nonattractive and nearly all trajectories escape from it. This kind of behavior is known as transient chaos [2, 8, 16].

In dynamical systems with transient chaos a complete analysis is possible for those showing a horseshoe map. It is the horseshoe which allows us to bound the location of the chaotic saddle in the phase space and thus the chaotic and escaping behavior.

In many real applications it might be desirable to control all the orbits starting close to the chaotic saddle while preserving the chaotic behavior. In mechanics, for example, preservation of transient chaos can prevent the appearance of undesired resonances. In lasers, it has been shown that maintaining transient chaos can help to avoid undesired intensity peaks. In engineering, it is known that the thermal pulse combustor operates chaotically, but when one tries to achieve high efficiency this can destroy the chaos and to cause the engine to flameout. In population dynamics the transition from transient chaos to periodicity is usually related with pathological situations (extinctions). These are examples that can be found in the techniques that have been proposed in recent years to achieve the goal of preserving transient chaos [3, 4, 6, 7, 14, 15, 18]. But two important issues must be addressed to solve this problem: The first one is the repulsive nature of the chaotic saddle, the second one is the environmental noise present in many physical situations, that typically makes the orbits escape even faster (although in some cases noise can slow down the escape process, see, [5]).

The partial control technique [20, 21] which is based in the application of small perturbations that will always keep the orbits inside the region Q, close to the chaotic saddle, was recently proposed to control transient chaos. The remarkable achievement of the partial control technique is that it allows us to control the system even when the amplitude of the corrections applied to the trajectories (the *control*) is smaller than the maximum deviation of the trajectories from their deterministic path due to the presence of environmental noise (the *noise*). The basic ingredient in order to obtain this somehow counterintuitive result is the use of certain sets referred to as *safe sets* [23] in Q. These sets have certain particular geometrical properties that are related to the typical stretching and folding action of the horseshoe-like mapping of Q. This advantageous control technique has been applied to well-known physical models [1, 17, 19, 20, 22, 23].

The main goal of this chapter is to develop a way of improving the partial control technique using what we call extended safe sets [11], instead of the safe sets. The extended safe sets are sets that we build making use of the escape times sets, and as we will show they also allow to keep trajectories bounded, with control smaller than noise offering some advantages over the safe sets. We will also explore in this chapter how does perform the partial control technique using safe sets, and extended safe sets and we will show which one is the best choice in different scenarios.

We also describe here an algorithm [12, 13] to find safe sets automatically: given the region in phase space from which trajectories escape and the value of the noise and control, safe sets are readily found. Basically we choose an initial phase space region and by iterations the algorithm eliminates those parts that do not hold with the condition required to be a safe set to finally obtain the desired safe set. We call it Iterative Sculpting Algorithm, as an analogy to removing material as in sculpting a statue.

This chapter is organized as follows: In Sec. 2 we describe the problem that we want to solve with our control strategy and we describe the system that we use in our explorations: the Hénon map. In Sec. 3 we review the main concepts of the safe sets. In Sec. 4 we define the escape time sets and explore their relation with the safe sets, and in Sec. 5 we show the conditions that extended safe sets need to fulfill. In Sec. 7 we review an algorithm that we have proposed recently to find safe sets in more general situations, i.e., situations where it is difficult to find horseshoe maps or when they are topologically very complex. Section 6 provides a numerical exploration of our control technique and a comparison between the results obtained with extended safe sets and safe sets. In Sec. 8 we draw the main conclusions of this work.

2. Problem Statement

We consider that the dynamics of the system considered is given by a map $\mathbf{p}_{n+1} = \mathbf{f}(\mathbf{p}_n)$, that can also be a Poincaré map, where $p_n \in \mathbb{R}^2$. We assume that the map f acts on a square Q like a horseshoe like-map, for details see [21]. This implies that nearly all the trajectories inside Q (except a zero measure set) escape from it after some iterations. On the other hand, the behavior inside the square Q is erratic due to the existence of this zero measure nonattractive set, the chaotic saddle.

As we said before, we consider systems with this kind of escaping dynamics $p_{n+1} = f(p_n)$ and also affected by noise. This is modeled here by adding at each iteration a random perturbation $\xi_n \in \mathbb{R}^2$ that we refer to as the *noise*, that is bounded by the constant ξ_0, $\|\xi_n\| \leq \xi_0$. Thus, the system to be controlled is $p_{n+1} = f(p_n) + \xi_n$. The effect of noise is that *all* the trajectories inside the square Q will now escape from it under iterations. In order to test our results we use here a noise with a uniform probability distribution but the control technique has to work for any other kind of distributions.

The ubiquity of this type of dynamical systems suggests that there are situations in which it might be desirable to control the system in order to keep the trajectories in the region where the horseshoe is defined (and thus far from an undesired attractor), although we might not need to determine exactly where the trajectory will go in Q. We call this type of control *partial control* of the system, and with this purpose, we can design a control strategy based on applying an accurately chosen *control* $\mathbf{u}_n$ each iteration, that we assume also bounded by a positive constant $||\mathbf{u}_n|| \leq u_0$, in such a way that the global dynamics is given by

$$\begin{cases} \mathbf{q}_{n+1} = \mathbf{f}(\mathbf{p}_n) + \boldsymbol{\xi}_n \\ \mathbf{p}_{n+1} = \mathbf{q}_{n+1} + \mathbf{u}_n, \end{cases} \tag{1}$$

so the control $\mathbf{u}_n$ depends on $\mathbf{p}_n$ and $\boldsymbol{\xi}_n$, as in other paradigmatic control methods [10].

A question that arises naturally here is, which is the value of u_0 needed to control the system, given the value of ξ_0. If the control is very strong, such that $u_0 > \xi_0$, it will not be difficult to find a strategy allowing to avoid escapes from the region Q. If $u_0 = \xi_0$, it might be possible. Remarkably, the partial control technique that we describe below allow us to achieve this goal even if the control is smaller than the noise, that is, if $u_0 < \xi_0$. As it will become clear later, this is due to the existence of certain sets inside the square Q: the *safe sets*.

An example of this type of dynamical system is the Hénon map *with an adequate choice of parameters*. The Hénon map defined as

$$\begin{cases} x_{n+1} = a - by_n - x_n^2 \\ y_{n+1} = x_n \end{cases} \tag{2}$$

is a paradigmatic system in nonlinear dynamics and for that reason we have chosen it, from now on, to show how the partial technique works.

We are interested here in the situation where $a = 6$ and $b = 0.4$. This is due to the fact that the Hénon map f acts like a horseshoe map on the square $Q \equiv [-4, 4] \times [-4, 4]$, as shown in Fig. 1.

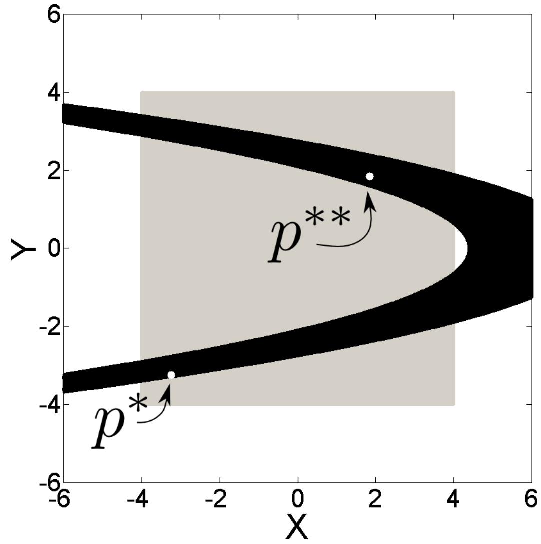

Fig. 1. The square $Q = [-4, 4] \times [-4, 4]$ (grey) and its image under the Hénon map $x_{n+1} = 6 - 0.4y_n - x_n^2$, $y_n = x_n$ (black). These are the parameters used in the chapter, for which the map acts like a horseshoe in Q. The two saddle fixed points p^* and p^{**} are shown.

The first implementation of the partial control technique required that the considered map acts on a square like a horseshoe map, i.e., that it satisfies the Conley-Moser conditions [21]. For these values of the parameters, the Hénon map satisfies these conditions, so we use it here both to illustrate and to numerically test our results.

In Fig. 1 we also show the two fixed points of this horseshoe-like map, p^* and p^{**}, the former will play later an important role (remember that every horseshoe map has associated two fixed points). The chaotic saddle responsible for transient chaos is shown in Fig. 2 and has been computed using the DYNAMICS software [9]. Due to the horseshoe mapping, this set is topologically equivalent to an intersection of two Cantor sets of vertical and horizontal lines, as expected. Thus, *nearly all* the points inside the square (except a zero measure set, the chaotic saddle and its stable manifold) escape from it under iterations. The dynamics inside the set is chaotic, but we have transient chaos due to its nonattracting nature for a typical trajectory starting inside Q.

3. Safe Sets

The partial control technique was originally defined, using safe sets, as the target of the control perturbations. We offer here a small review of their

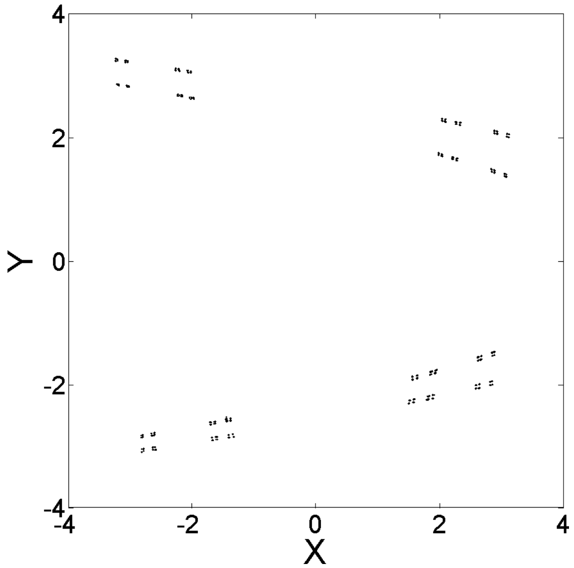

Fig. 2. Chaotic saddle for the Hénon map $x_{n+1} = 6 - 0.4y_n - x_n^2$, $y_n = x_n$.

main properties that would allow us to introduce the extended safe sets in a natural way.

Basically the safe sets are a set of curves that are placed in the square where is located the horseshoe map. We can see the form of different S^k safe sets in the Hénon map with escaping dynamics in Fig. 3. As we can see in that figure it is possible to group these curves using a specific order that we denote as k.

Bearing this graphical idea of safe sets in mind we can already define the main properties for the curves that are part of S^k (the safe set of order k.):

- S^k consist of 2^k vertical curves.
- Any vertical curve of S^k has two *adjacent* vertical curves of S^{k+1} closer to it than any other curve of S^k.
- The maximum distance between any of the 2^k curves of S^k and its two adjacent curves of S^{k+1}, denoted as δ_k, goes to zero as $k \to \infty$.
- All the curves of a S^k set can be grouped in pairs. Each pair would be composed of the two closest curves.

The algorithm to find the adequate safe set in a system with escapes make use of the inverse of the horseshoe map. The procedure consists of making backward iterations (applying the inverse map) of a straight line centered in the square as we can see in Fig. 4. Making these iterations and computing their intersections with the square, we easily obtain all the curves that are part of a particular S^k, where the order of the safe set corresponds to the number of iterations. Thus, it is possible to obtain the adequate safe set S^k that makes possible to avoid escapes even with a control smaller than the noise.

To properly quantify the properties of the partial control technique we need to define three parameters to establish the condition of a control smaller than the noise for a particular S^k:

- The first is the curve, ζ_i, that represents the middle distance in a particular pair of curves for a given S^k set.
- The second is the distance, $\delta_{max(i)}$, from the two points of that pair whose distance to ζ_i is maximum.
- The third is the distance, $\delta_{min(i)}$, from the two points of that pair whose distance to ζ_i is minimum.

We can visualize these parameters in Fig. 5 where we have plotted all of them for the two pairs of curves present in the S^2 set.

If we label as δ^*_{max} as the largest $\delta_{max(i)}$ among all the pairs of curves, then it is easy to understand that the condition to have a control smaller than the noise is that

$$\xi_0 > \delta^*_{max}. \tag{3}$$

When this condition is satisfied, we can also compute the maximum control present in the system if we consider δ^*_{min} as the smallest of all the δ_{min} among all the pairs of curves. Then we have that

$$u_0 = max\{\delta^*_{max}, \xi_0 - \delta^*_{min}\}. \tag{4}$$

4. Escape Time Sets

In order to define what is a extended safe set, it is necessary to review the idea of escape time of a trajectory starting in a region Q where exists a escaping dynamics. A point has a *escape time* n if the number of iterations that are needed to leave Q is n.

Using the idea of escape time it is possible to group all the points in the square Q in what we call T^n sets. We denote as T^n sets the sets of points in a square with a horseshoe dynamics, that stay inside the square under n iterations or more, that is:

$$T^n = \{p \in Q / f^n(p) \in Q\}. \tag{5}$$

This means that all the points with escape time n, $n+1$, $n+2\ldots$ etc lie inside T^n. For example, T^3 would be composed of all the points with escape time 3, 4, 5 and so forth into infinity.

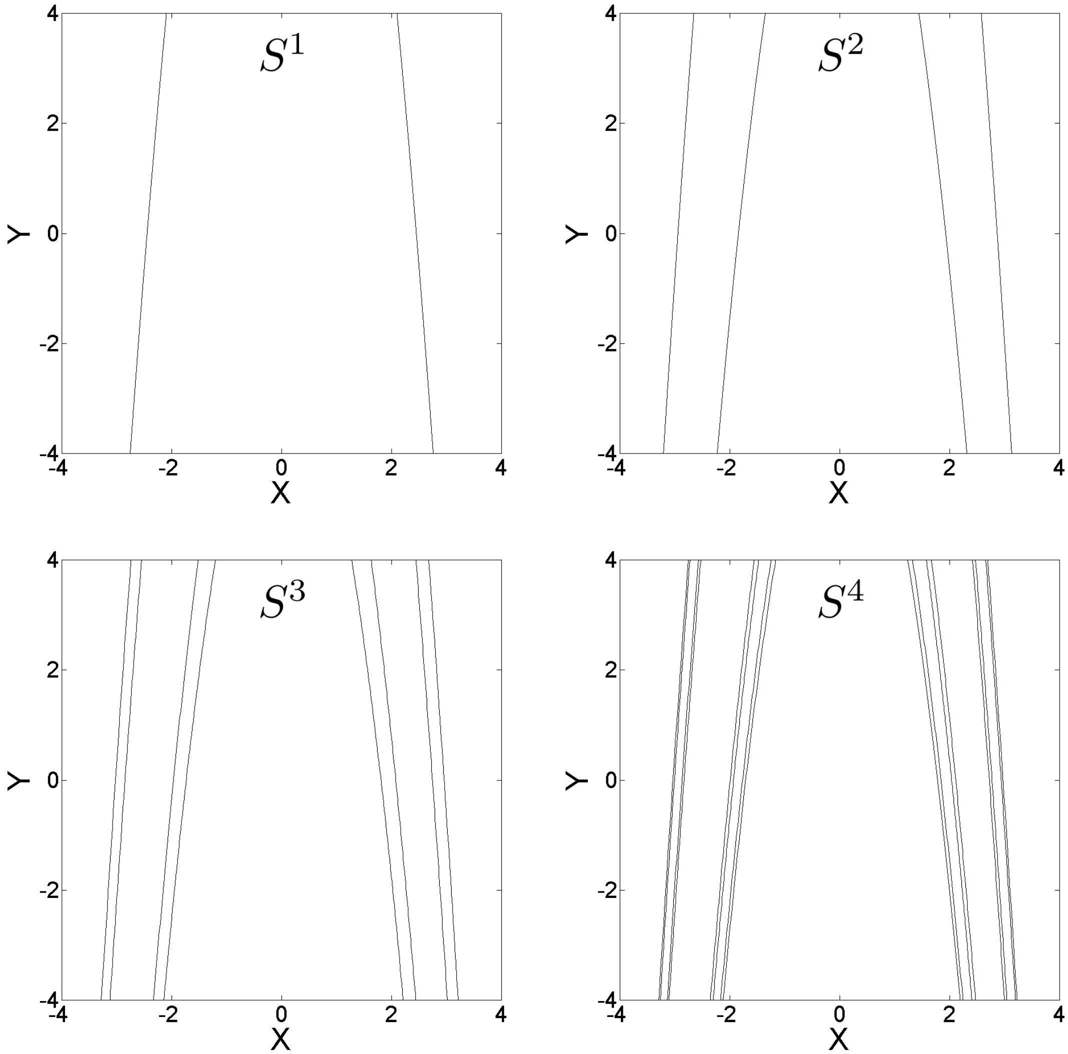

Fig. 3. The sets S^1, S^2, S^3 and S^4 that are the result of computing the corresponding preimage of S^0, that would be the vertical segment splitting the square into two equal rectangles. The set S^1 consists of a pairs of curves, the set S^2 has two pairs of curves, the set S^3 consists of four pairs of curves, and so on. Their geometrical properties are key for application of the partial control technique.

The appearance of the T^n sets, for a horseshoe-like map, are quite similar to curved strips as we can see in Fig. 6 for the T^3 set. With this graphical idea of the T^n sets it is possible to establish the following properties for the T^n sets:

- The T^n set consist of 2^n vertical strips.
- Inside each T^n we have all the sets of higher order, i.e $T^{n'>n} \subset T^n$.
- As the order n increases the width of the strips of the size of T^n sets decreases.
- All the strips of a T^n set can be grouped in pairs. Each pair would be composed of the two closest strips.

We also know that the safe sets of order n (S^n) are by definition inside the corresponding T^n set. This is due to the fact that on the one hand S^0 has escape time $n = 0$ and that on the other hand any S^n set is computed using n preimages of S^0. So it is quite obvious that, for example, S^1 has escape time $n = 1$ because in one iteration all their points are mapped to S^0 that still stays in Q. For S^2 we have that in one iteration we go to S^1 and in two iterations to S^0 so it has escape time $n = 2$. Thus it is clear that

$$S^n \subset T^n. \tag{6}$$

Considering this, it might be possible that some of the points inside T^n (not only those of S^n) could be used as "extended safe sets" (since they contain more points than the normal safe sets). However, not all the points of the T^n sets are valid to achieve

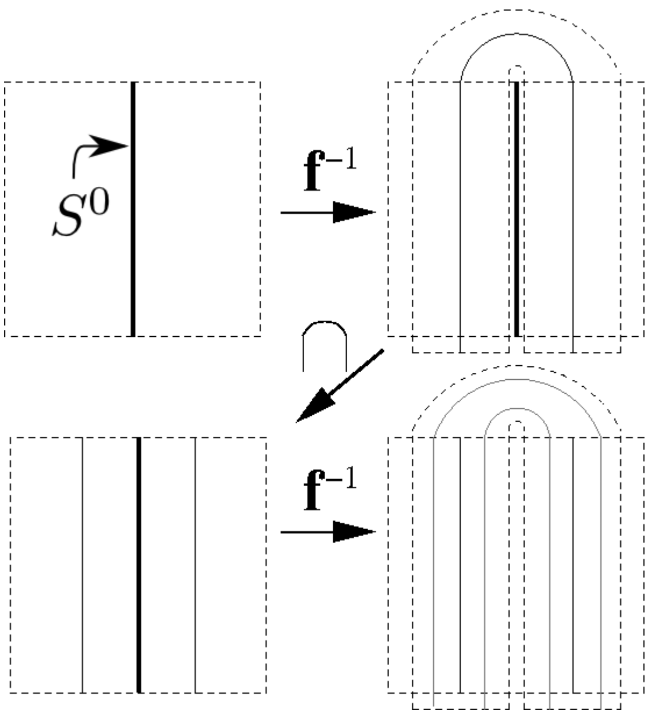

Fig. 4. Basic action to generate inductively the safe sets in a map showing a horseshoe-like behaviour in the phase space. We begin with a vertical segment (S^0) that splits the square in two equal halves and we compute the preimages of that line in Q. The intersections of the kth preimage of S^0 with Q gives the set S^k, that consists of 2^k curves.

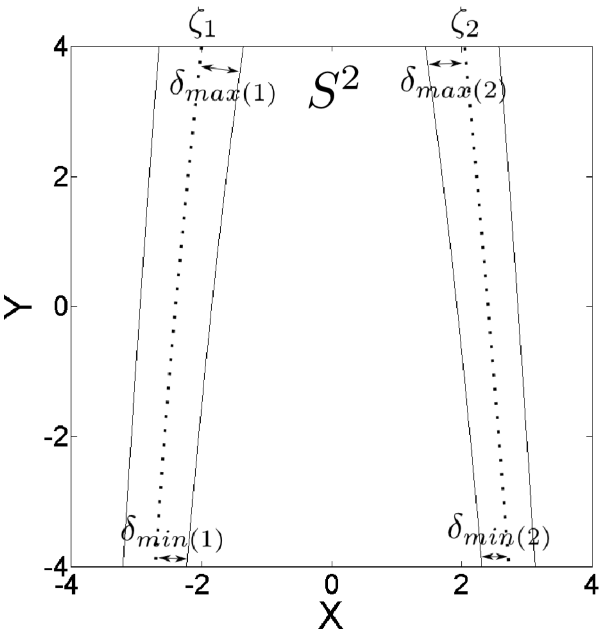

Fig. 5. These are the parameters needed in the partial control technique using the S^2 sets. The curves ζ_1 and ζ_2 are the curves whose points are at the same distance from a curve of each pair of curves of S^2. We can also see that $\delta_{max(i)}$ is the maximum distance from each pair of curves of S^2 to the curves ζ_i as well as $\delta_{min(i)}$ is the minimum distance.

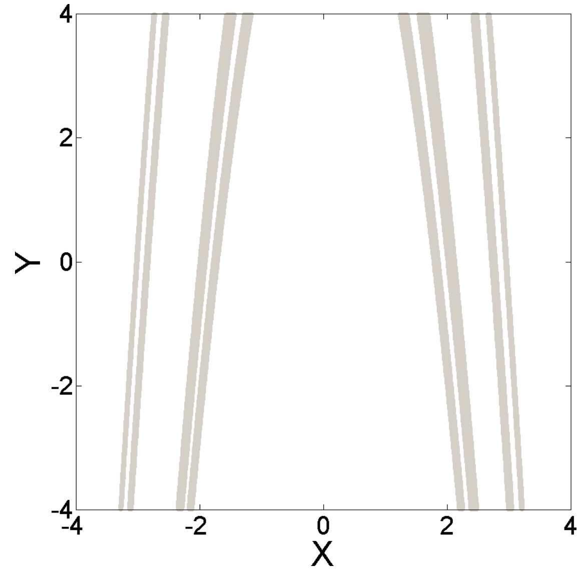

Fig. 6. The escape time set T^3, i.e., the set of points that escape from Q after 3 or more iterations. It consists of four pairs of strips, and it is reminiscent of S^3.

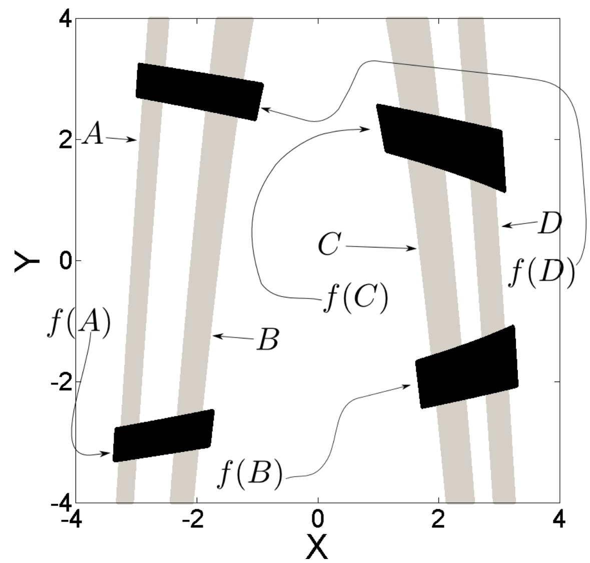

Fig. 7. In this picture we can see in grey the set of points of T^2 (the set of points that stay in the square under 2 or more iterations). We have also plotted in black the images of those points under one iteration. Due to the fact that the images of the strips do not lay between the pairs of strips, it is impossible to use the T^2 sets with a control smaller than the noise.

a control smaller than the noise, as we can see in Fig. 7. In this figure, we can see the forward iterate, in black, of the strips corresponding to the set T^2 (number of points that stay in the square under two iterations or more) which appear colored in grey. We can see in this figure that not all the parts of the forward iterates of the strips lie between some *"pair of strips"*, and this is true for other T^n sets. Thus, if we use the set T^n instead of a safe set in S^n in the partial control strategy it will force us for the worst conditions of noise to apply a control higher than the noise, so it would not be possible to keep trajectories inside Q with $u_0 < \xi_0$.

5. Obtaining Extended Safe Sets from Escape Time Sets

As we said above, clearly the escape time sets T^n are not a good substitute for safe sets S^n in the partial control technique. The next question would be: which points on T^n have to be discarded so that the partial control strategy can be applied with $u_0 \leq \xi_0$? In this section we address this question. We describe here the *maximal extended safe set* E^n_{max}, that is a subset of T^n that can be used in the partial control technique so that trajectories can be kept bounded with $u_0 = \xi_0$. From these sets we can easily define the *extended safe sets* E^n, and for any E^n we show that there is a ξ_0 so that trajectories can be kept inside Q with $u_0 < \xi_0$.

Recall the fixed point p^* shown in Fig. 1. We call $W^s(p^*)$ its stable manifold, and $W^s_L(p^*)$ the vertical curve that is a piece of $W^s(p^*)$ inside Q containing p^*. Consider now the set T^1 and the four vertical curves of $f^{-2}(W^s_L(p^*)) \cap Q$ shown in Fig. 8. We call E^1_{max} the set resulting of "cutting" the two strips of T^1 into two thinner strips as these four curves indicate. The strips of E^1_{max} are mapped as shown in Fig. 8. This is due to the horseshoe mapping and to the fact that points in the stable manifold map into points of the stable manifold under f. This is the limit point of the "good mapping" that we are searching for: the image of each strip of E^1_{max} falls into the space between the pair of strips of E^1_{max}.

Considering this, we define inductively

$$E^{n+1}_{max} = f^{-1}(E^n_{max}) \cap Q = f^{-n}(E^1_{max}) \cap Q. \quad (7)$$

Clearly the set E^n_{max} consists of 2^n strips, that also can be grouped in 2^{n-1} pairs of curves from left to right. Note that by definition it will be contained in T^n. Furthermore, it can be seen that the curves that bound each vertical strip of E^n_{max} are pieces of the stable manifold (since preimages of points of the stable manifold also belong to the stable manifold). These sets will reproduce the good kind of mapping observed for E^1_{max}: The image of each strip of E^n_{max} falls in the space between each pair of strips of E^n_{max}. This is shown for example in Fig. 9 for the set E^2_{max}. Using the geometrical considerations provided above, we can see that by using the sets E^n_{max} instead of S^n in the partial control strategy trajectories can be kept inside Q with $u_0 = \xi_0$.

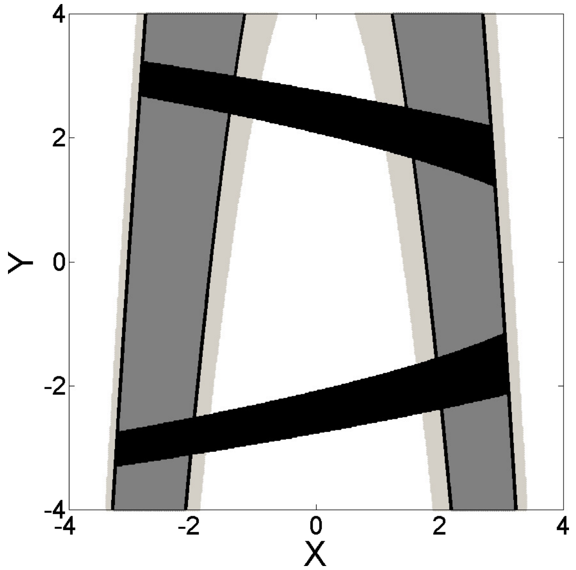

Fig. 8. The set T^1 (light grey) is subdivided using four pieces of the stable manifold of p^* (black curve) so we obtain the maximal extended safe set E^1_{max} (grey). The image of E^1_{max} under f is shown (black), each of its two pieces falls in the space between the pair of strips of E^1_{max}.

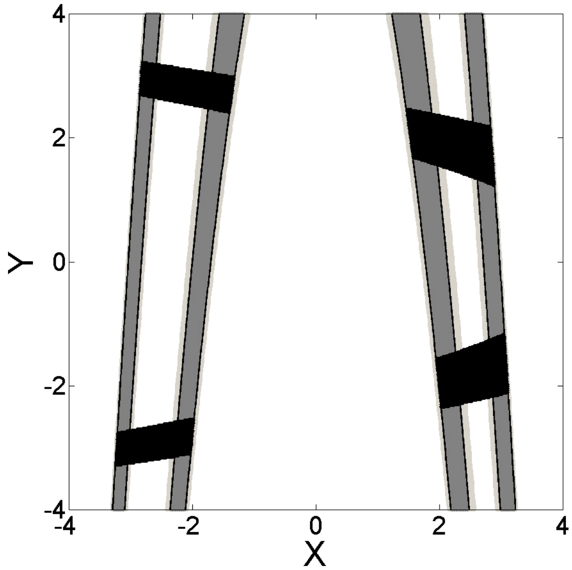

Fig. 9. The set T^2 (light grey) is cut using eight pieces of the stable manifold of p^* (black curve) and gives rise to the maximal extended safe set E^2_{max} (grey). The images of each strip of E^2_{max} under f are shown (black), they fall in the space between each pair of strips of E^2_{max}.

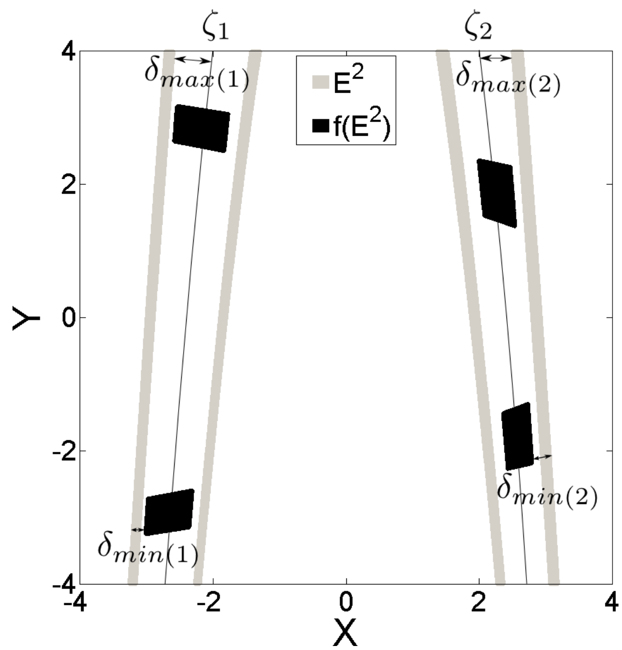

Fig. 10. In this figure we can see an extended safe set E^2 and the parameters needed in the partial control technique using the extended safe set E^2.

With these elements in mind, we can define the *extended safe sets* E^n as follows: An extended safe set E^n is a set of 2^n vertical strips, each of them inside a different strip of E^n_{max}, so that their vertical bounds do not intersect with the vertical bounds of the strips of E^n_{max}. Thus, a extended safe set is obtained when the width of all the strips of the extended safe sets is reduced. If we take zero-width strip we would obtain safe sets as the ones described in Sec. 3. An example of an extended safe set E^2 obtained from E^2_{max} is shown in Fig. 10.

Before concluding this section, we would like to point out that the same procedure that has been carried out with the Hénon map to obtain the extended safe sets could be repeated with the same level of difficulty for any topologically equivalent dynamical system, i.e., a dynamical system acting as a horseshoe on a (topological) square Q. It is important to notice that in order to find the extended safe sets one needs to find the chaotic saddle, a square enclosing it, the escape time sets and the stable manifold of the fixed point of the horseshoe. These can be calculated using time series of the system, and do not require to know exactly the form of the map f. Thus, we consider that this is a first advantage in the use of extended safe sets from the point of view of its applicability.

6. Simulations

Here we show the results that we have achieved by using this new technique. We have carried out different kind of simulations to check that this new technique works as we expected and how it performs for different levels of noise.

We have use as a basis to construct our extended safe sets the T^2 and T^3 escape sets. Of course not all the points inside T^2 or T^3 are valid to keep the condition of a control smaller than the noise. As we know from the 3th step of our strategy, we should remove at least the points that lie beyond the most external overlapping between the stable manifold of the saddle point and the escape time strips. In our simulation we have been a little bit more aggressive and we have removed all the points that had an image on some strip too. Those have been the E^n sets in our simulation.

In our first simulation, we have considered that the maximum level of noise present is $\xi_0 = 0.25$ and the area from where we want to avoid the escape is the square Q, where is located the chaotic saddle. In this situation it is clear that the most effective extended safe set that we can use here is E^3. We have carried out a succesful simulation of 1000 iterations with a control smaller than the noise, as can be seen in Fig. 11.

We have carried out another kind of simulation to compare the technique developed in [23] and the technique proposed here. As we expected, there are some values of the noise in which the S^n are a better strategy in terms of maximum control and

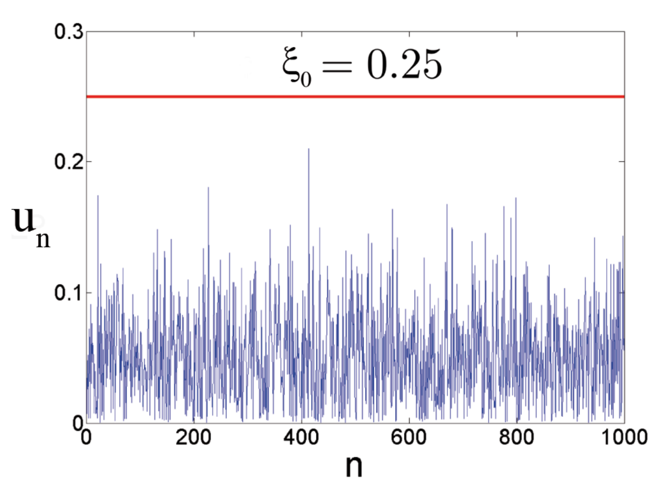

Fig. 11. Here we can see a simulation of 1000 iterations in $x_{n+1} = 6 - 0.4y_n - x_n^2; y_{n+1} = x_n$ where the maximum amplitude of noise is $\xi_0 = 0.25$. As we can check the control applied in each iteration u_0 is smaller than the noise.

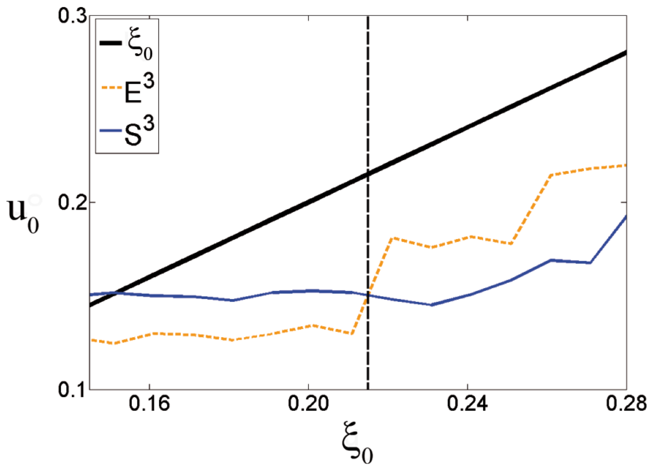

Fig. 12. Here we show a comparison of the safe sets technique and the extended safe sets technique for different values of noise. Here we have carried out series of 2000 iterations and we have computed the maximun control needed for different values of noise. As we can see in the graphic there are some values of noise where the extended safe sets technique is most efficient and others where the safe sets technique is the most efficient. It is also possible to see here that the extended safe set used in the simulation is going to be the optimum for a continuous uniform distribution of noise with a maximun amplitude of roughly 0.19.

others in which the E^n sets are a better solution. This is what we can see in Fig. 12. We can also see that there exists a transition point that indicates the limit where both techniques are more efficient.

7. An Algorithm to Automatically Compute the Safe Sets

The above methodology to find safe sets works in situations in which we have a map that acts like a horseshoe in a topological square Q. But it also provided us the necessary insights to find safe sets in more general settings. We sketch here an algorithm [12] that we have designed in order to find safe sets (and thus apply our partial control technique) in more general situations.

In order to find safe sets in more general settings, one has to think first which is the basic property of a safe set. Given a phase space region Q from which a trajectory escapes under a map f, and where there is a noise ξ bounded by ξ_0, and where it is possible to apply perturbations u also bounded by u_0 before each iteration, a safe set S is a set contained in Q that satisfies the following property:

$$\max_{p\in S, |\xi|\le\xi_0} d(f(p)+\xi, S) = u_0 < \xi_0, \tag{8}$$

where $d(\cdot,\cdot)$ represents the distance between a point and a set of points.

We can see that the safe sets found for horseshoe maps in the previous section satisfy this condition. For this reason, it is not difficult to see why any set satisfying equation (8) works as a suitable safe set. Assume that p is any point on S. The map takes it to $f(p)$, but it will be deviated by the noise to $q = f(p) + \xi$. Equation (8) implies that no matter which point we consider and the value of ξ, as long as it is bounded by ξ_0, the distance between $q = f(p) + \xi$ and S will be *always* smaller or equal to u_0, and smaller than ξ_0. Thus, with an adequate control u smaller than ξ_0, we can put $f(q) + \xi + u$ on a point of the safe set and this can be repeated forever. Of course, the safe sets for horseshoe maps described previously satisfy this property.

Making use of this idea, we have designed an algorithm [12] that is able to find safe sets in any situation, provided that the phase space region from which trajectories escape Q, the value of the noise ξ_0 and the value of the control u_0 are known. The basic idea of this algorithm is to start with an initial set of points and to take recursively only points that are mapped in a good way (as prescribed by Eq. (8)), until converging to the desired safe set. We have also studied the dynamics of partial control by using the safe sets obtained with this new algorithm [13].

The procedure is then the following: we start with a given set of points $S(0)$ inside the region Q (it is possible to chose as $S(0)$ a grid of points on Q). Then, we pick as points of $S(1)$ only the points $p \in S(0)$ that satisfy the following condition for all ξ such that $|\xi| \le \xi_0$:

$$d(f(p)+\xi, S(0)) \le u_0. \tag{9}$$

To obtain S(2), S(3), etc... we apply the general rule: a point $p \in S(n)$ belongs to $S(n+1)$ if it satisfies that for all $|\xi| \le \xi_0$:

$$d(f(p)+\xi, S(n)) \le u_0. \tag{10}$$

Thus, we generate iteratively certain sets

$$S(0) \supset S(1) \supset S(2) \supset S(3) \supset \cdots \supset S(n). \tag{11}$$

The resulting set $S(\infty)$, when $n \to \infty$, unless it is not an empty set, is a safe set.

8. Conclusions and Discussion

In this chapter we have shown that it is possible to apply the partial control technique using certain sets, the extended safe sets, that are deeply related with the escape time sets, in a square Q where a horseshoe map exists. The notion of extended safe sets generalizes the notion of safe sets when we use these new sets in the partial control strategy, to keep trajectories bounded in the square, with a control smaller than the noise. The procedure to obtain extended safe sets from escape time sets has been described and it implies the cut of the escapes times using the stable manifold of a saddle fixed point in a horseshoe map (remember that every horseshoe map has associated two fixed points).

From an experimental point of view the use of extended safe sets, being a nonzero measure set, is advantageous. Having an extended surface, it is easier to place trajectories on it without making an error than when dealing with zero measure sets as the safe sets. On the other hand, by construction they are computed from the escape time sets and using the stable manifold of a fixed point as a guide, and this information can be inferred from time series of the system.

Throughout this chapter we have assumed that our control has no errors, that is, that at each iterate we can place the trajectory exactly where we want. However, this is not a critical assumption. As with safe sets, it is possible to keep the condition $u_0 < \xi_0$ using extended safe sets also if we have small control errors [21], i.e., even if at each iteration trajectories are not placed exactly on the extended safe sets. This tolerance to errors depends basically on the value of u_0 needed in absence of errors, the value of ξ_0 and on the expansiveness of the map f, which somehow tells how much are we penalized if we do not apply exactly the required control. To provide an analytical estimate of such tolerance, though, is complicated. However, due to the fact that extended safe sets are "thicker" than the safe sets, we expect that the tolerance for the former is bigger.

We have also described a general algorithm that allows to find safe sets for any nonlinear dynamical system in order to apply the partial control technique. Such safe sets can be found inside the regions from which trajectories escape after having some complex dynamical behavior, and their existence guarantees that trajectories can be kept inside that region with a control smaller than noise. Our algorithm is of a general nature, in the sense that it can be applied for any map or flow. The algorithm eliminates those parts from an initial phase space region that do not hold with the condition of the safe set to finally obtain the desired safe set. We call it Iterative Sculpting Algorithm, as an analogy to removing material as in sculpting a statue.

Finally, we want to emphasize that our analysis reveals the deep relation existing between the escape time sets and these extended safe sets, so we believe that any algorithm implemented in order to detect extended safe sets should make use of this relation: first, searching for the different escape time sets and then discarding the points that are not useful. As we said, this can be helpful both for an experimental detection of safe sets that can allow to obtain partial control with $u_0 < \xi_0$ as well as in generalizations of the partial control technique to dynamical systems in higher dimensions.

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