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Effect of squeezing and Planck constant dependence in short time semiclassical entanglement

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Abstract. In this paper, we investigate into the short time semiclassical entanglement of a general class of two-coupled harmonic oscillator system that includes additional nonlinear terms in the potential of the form $\lambda \mathbf{x}^m \mathbf{y}^n$, such that the sum of the degree m and n equals to a fixed constant. An analytical expression of the short time linear entropy is derived and it shows a clear relationship between the single mode squeezing and the entanglement dynamics. In addition to that, our theoretical analysis has shown that the short time semiclassical entanglement entropy displays a dependence on the Planck constant $\hbar$ of the form $\hbar^{m+n-2}$ for this class of systems. By applying our results to the linearly coupled harmonic oscillator, the Barbanis-Contopoulos, the Hénon-Heiles and the Pullen-Edmonds Hamiltonian, we have found a good correspondence between the numerical and analytical results in the short-time regime. Interestingly, our results have demonstrated both analytically and numerically that an appropriate manipulation of initial squeezing can have the significant effect of enhancing the short time semiclassical entanglement between the two subsystems.

1 Introduction

The study of short time entanglement has importance in uncovering the underlying mechanisms where a fast rate of entanglement production occurs. Such a study has been performed by Angelo and Furuya [1] through the analysis of the semiclassical limit of the entanglement in the Dicke model and the coupled Kerr oscillator. In a similar context, Žnidarič and Prosen [2] have analyzed the generation of entanglement in regular system by using the echo operator. In addition, by performing a semiclassical analysis on entanglement generation within bipartite quantum system, Jacquod [3] had found that for the short-ranged interaction potential, the entanglement production is exponentially fast in chaotic systems, while algebraic in regular systems. In many other related works, the entanglement production at the initial short time is found to be a good indicator of the regular-to-chaotic transition [4–9].

From a practical point of view, entanglement is known to be an important resource for the purpose of quantum information processing [10–12]. For example, the ability to exert control on the rate of entanglement generation would enable the secure transmission of information against the most general coherent attacks [13]. An extremely useful approach to generate continuous variable entanglement is that of two-mode squeezing. If we were to perform single-mode squeezing prior to the two-mode squeezing, it has been shown via diverse quantum systems that the generation of entanglement can be enhanced. This notion has been demonstrated in the Jaynes-Cummings model where a stronger entanglement between a two-level atom and an electromagnetic field mode is attained by employing a squeezed state rather than a coherent state as the initial photon state [14]. Note that the enhancement only arises when the initial state of the field mode is sufficiently squeezed. Similar threshold has also been observed in systems of coupled harmonic oscillators [15]. In these cases, the maximum entanglement is observed to grow steadily with an increase in the initial squeezing parameter beyond the threshold. In fact, the enhancement in entanglement can also be studied by performing unequal single-mode squeezing separately on the two field modes [16]. Notably, entanglement is found to persist even in a decohering environment with high temperature when the normal modes are squeezed [17]. In addition to that, Wang and Sanders [18] have analyzed symmetric multiqubit states and they have found a clear relationship between spin squeezing and pairwise entanglement. More recently, Beduini and Mitchell [19] have extended the results of reference [18] to optical fields and they have found a spin-squeezing inequality for photons.

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The consideration of these works has led us to the question of whether initial squeezing has the effect of enhancing the rate of entanglement production in the short time regime. In this paper, we aim to answer this question both analytically and numerically through quantifying entanglement by the linear entropy and studying its dynamics. Moreover, we will explore the general result of $\hbar$ dependence in the short time regime. We shall consider a general interaction potential of the form $V_\lambda = \lambda \mathbf{x}^m \mathbf{y}^n$ since it is applicable to a wide range of nonlinear chaotic oscillators. In particular, we investigate into the linearly coupled harmonic oscillators, and the Barbanis-Contopoulos [20,21], the Hénon-Heiles [22–28] and the Pullen-Edmonds Hamiltonian [29–32] in this paper. The interest in exploring these systems lies in their intrinsic rich and contrasting classical and quantum dynamical behaviour. In addition, there is as yet no research being performed to understand the relationship between entanglement dynamics and the initial squeezing effect in the short time regime for these systems. For this purpose, we shall employ the tensor product of a squeezed vacuum state to initiate the entanglement dynamics. Our restriction to the consideration of initial squeezed vacuum state instead of generic squeezed coherent state results from analytical tractability in the derivation of our final results. In summary, our main focus in this paper is to analyze the relationship between initial squeezing and the short time entanglement dynamics as well as its $\hbar$ dependence in the short time regime.

Our paper is organized as follows. First, we perform mathematical analysis on the short time entanglement for the case of a general interaction potential $V_\lambda = \lambda \mathbf{x}^m \mathbf{y}^n$ when the initial state is a tensor product of squeezed vacuum state. This allows us to determine the analytical expression of the linear entropy of the resulting short time entanglement dynamics. Then, we apply these results to the linearly coupled harmonic oscillator, and the Barbanis-Contopoulos, the Hénon-Heiles and the Pullen-Edmonds Hamiltonian, where good agreement is found between our numerical and analytical results.

2 Model description

We consider a general two-dimensional classical Hamiltonian of the form:

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{2}\left(x^2 + y^2\right) + \lambda x^m y^n. \tag{1}$$

In the quantum case, the corresponding Schrödinger equation can be written as

$$i\hbar \frac{\partial}{\partial t}\psi(\mathbf{x},\mathbf{y},\mathbf{t}) = \frac{-\hbar^2}{2m}\nabla^2\psi(\mathbf{x},\mathbf{y},\mathbf{t}) + V(\mathbf{x},\mathbf{y})\psi(\mathbf{x},\mathbf{y},\mathbf{t}), \tag{2}$$

where $V(\mathbf{x},\mathbf{y}) = \frac{1}{2}(\mathbf{x}^2 + \mathbf{y}^2) + \lambda \mathbf{x}^m \mathbf{y}^n$ is the two-dimensional potential with λ being the coupling constant. For analytical purposes, we shall split the Hamiltonian into a sum of uncoupled harmonic oscillator Hamiltonian and an interaction potential. Hence, the Schrödinger equation can be rewritten as

$$i\hbar \frac{\partial}{\partial t}\psi(\mathbf{x},\mathbf{y},\mathbf{t}) = \left(\hat{H}_0 + \hat{V}_\lambda\right)\psi(\mathbf{x},\mathbf{y},\mathbf{t}), \tag{3}$$

where $\hat{H}_0 = \frac{-\hbar^2}{2m}\nabla^2 + \frac{1}{2}(\mathbf{x}^2 + \mathbf{y}^2)$ and $\hat{V}_\lambda = \lambda \mathbf{x}^m \mathbf{y}^n$.

The time evolution of the quantum state $\psi(\mathbf{x},\mathbf{y},\mathbf{t})$ is given by

$$\psi(\mathbf{x},\mathbf{y},\mathbf{t}) = \hat{U}(\mathbf{t})\psi(\mathbf{x},\mathbf{y},\mathbf{0}), \tag{4}$$

where the time evolution operator is given by:

$$\hat{U}(\mathbf{t}) = \exp\left(\frac{-i\mathbf{t}}{\hbar}(\hat{H}_0 + \hat{V}_\lambda)\right).$$

In our numerical computations, the time evolution of the wave function is performed by means of the second-order split operator technique. Feit et al. [33,34] have done a detailed analysis on the time evolution of the wave packet in the Hénon-Heiles potential and we have followed their approach in this work. By using the general quantum Hamiltonian given by equation (3), we now proceed to study the short time continuous variable entanglement in the semiclassical regime both analytically and numerically.

First, we write a pure continuous bipartite state as follows:

$$|\psi\rangle_{12} = \int \psi(\mathbf{x},\mathbf{y})|\mathbf{x}\rangle|\mathbf{y}\rangle dxdy, \tag{5}$$

where $|\mathbf{x}\rangle$ and $|\mathbf{y}\rangle$ are the continuous basis representation of the position operators of the first and second particle, respectively. The reduced density function of the first subsystem ρ_1 can be obtained by summing over the second field mode, and it can be expressed in terms of the bipartite wave function $\psi(\mathbf{x},\mathbf{y})$, i.e,

$$\rho_1(\mathbf{x},\mathbf{z}) = \int \psi(\mathbf{x},\mathbf{y})\psi^*(\mathbf{z},\mathbf{y})dy, \tag{6}$$

where $\rho_1(\mathbf{x},\mathbf{z})$ is the reduced density function of the first subsystem in the continuous position basis representation.

To quantify the continuous variable entanglement, we use the linear entropy of entanglement based on numerical methods proposed in references [35,36]. The linear entropy of entanglement $\delta(t)$ is defined as

$$\delta(t) = 1 - \sum_i \lambda_i^2, \tag{7}$$

where λ_i are the eigenvalues of the Hermitian kernel $\rho_1(\mathbf{x},\mathbf{z})$. These eigenvalues are numerically computed from the Fredholm type I integral equation of $\rho_1(\mathbf{x},\mathbf{z})$, which is given by

$$\int \rho_1(\mathbf{x},\mathbf{z})\phi_i(\mathbf{z})dz = \lambda_i\phi_i(\mathbf{x}), \tag{8}$$

where λ_i are the eigenvalues with the corresponding Schmidt eigenfunctions $\phi_i(\mathbf{x})$. Note that there exists another definition of the linear entropy in terms of the continuous basis representation, and it can be written as:

$$\delta(t) = 1 - \mathrm{Tr}\left({\rho_1}^2\right) = 1 - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho_1(\mathbf{x},\mathbf{z})\rho_1(\mathbf{z},\mathbf{x})dxdz. \tag{9}$$

This definition is well suited for analytical calculation of the linear entropy. Hence, for numerical computation, we shall use the definition of linear entropy given by equation (7) while for theoretical analysis, we use the definition given by equation (9).

3 The squeezed coherent state

Since we are treating our system in terms of the continuous position basis, we have to represent the squeezed coherent state in terms of the coordinate representation. A year after the discovery by Rai and Mehta's on the coordinate representation of the squeezed coherent state [37], Hong-Yi and VanderLinde [38] have found another analytical expression of the same wave function. Nonetheless, we prefer the one proposed by Møller et al. [39] due to the relative simplicity of the wave function. These authors have followed Hollenhorst [40] and Caves's [41] definition of the displacement and squeezing operators, which is given by

$$\hat{D}(\alpha_k) = \exp\left(\alpha_k \hat{a_k}^\dagger - \alpha_k^* \hat{a_k}\right) \tag{10}$$

and

$$\hat{S}(\zeta_k) = \exp\left(\frac{1}{2}\zeta_k \hat{a_k}^{\dagger^2} - \frac{1}{2}{\zeta_k}^* \hat{a_k}^2\right). \tag{11}$$

The squeezed coherent state is thus defined as

$$|\alpha_k, \zeta_k\rangle = \hat{D}\left(\alpha_k\right) \hat{S}\left(\zeta_k\right) |0\rangle. \tag{12}$$

Here $\alpha_k = |\alpha_k| e^{i\phi_k}$ and $\zeta_k = r_k e^{i\theta_k}$ are complex numbers. α_k is related to the phase space variables (q_k, p_k), which is given by

$$\alpha_k = \frac{1}{\sqrt{2\hbar}}\left(q_k + ip_k\right), \tag{13}$$

where $k = 1, 2$, respectively. According to Møller et al. [39] the squeezed coherent state in the position basis can be written as

$$\begin{aligned} \psi(\mathbf{x}, \alpha_k, \zeta_k) = &\left(\frac{1}{\pi\hbar}\right)^{1/4} \left(\cosh r_k + e^{i\theta} \sinh r_k\right)^{-1/2} \\ &\times \exp\left\{-\frac{1}{2\hbar}\left(\frac{\cosh r_k - e^{i\theta}\sinh r_k}{\cosh r_k + e^{i\theta}\sinh r_k}\right)\right. \\ &\left.\times\left(\mathbf{x} - q_k\right)^2 + \frac{i}{\hbar} p_k \left(\mathbf{x} - q_k/2\right)\right\}. \end{aligned} \tag{14}$$

In our theoretical and numerical analysis, we shall use the tensor product state of this wave function with $\alpha_k = 0$ to study the quantum entanglement dynamics for different squeezing parameter values.

4 Short time entanglement of the squeezed vacuum

In order to calculate the time evolution of the wave function, we shall make use of the Zassenhaus formula [42], which is the dual of the Campbell-Baker-Hausdorff formula. Zassenhaus formula can be written as

$$e^{t(X+Y)} = e^{tX}\ e^{tY}\ e^{-\frac{t^2}{2}[X,Y]} \ldots \tag{15}$$

The short time evolution of the wave function is obtained through the evaluation of Zassenhaus formula followed by truncating all the higher order terms in t, with the time evolution operator expressed in the following way:

$$\begin{aligned} \hat{U}(t) &= \exp\left(\frac{-i\Delta t}{\hbar}\hat{H}\right) \\ &\approx \exp\left(\frac{-i\Delta t}{\hbar}\hat{V}_\lambda\right) \exp\left(\frac{-i\Delta t}{\hbar}\hat{H}_0\right). \end{aligned} \tag{16}$$

Consider the squeezed coherent wave function associated with the $\mathbf{x}$ variable in the position basis $\psi(\mathbf{x}, \alpha_1, \zeta_1)$, as given in equation (14). Here, α_1 is the center of the Gaussian wave packet and ζ_1 is the squeezing parameter. Similarly, we consider the squeezed coherent wave function associated with the $\mathbf{y}$ variable in the position basis which is $\psi(\mathbf{y}, \alpha_2, \zeta_2)$. Hence, the tensor product state at time $t = 0$ can be written as

$$\psi(\mathbf{x}, \mathbf{y}, \mathbf{0}) = \psi(\mathbf{x}, \alpha_1, \zeta_1)\psi(\mathbf{y}, \alpha_2, \zeta_2). \tag{17}$$

Now, we treat the tensor product of the squeezed vacuum state by taking $\alpha_1 = \alpha_2 = 0$. Hence, the initial wave packet at time $t = 0$ is given by:

$$\psi(\mathbf{x}, \mathbf{y}, \mathbf{0}) = \left(\frac{1}{\pi\hbar}\right)^{1/2} N_1 N_2 \exp\left(\frac{-\left(\mathbf{x}^2/\eta_1 + \mathbf{y}^2/\eta_2\right)}{2\hbar}\right), \tag{18}$$

where

$$N_k = \left(\cosh r_k + e^{i\theta_k}\sinh r_k\right)^{-1/2}$$

and

$$\eta_k = \left(\frac{\cosh r_k + e^{i\theta_k}\sinh r_k}{\cosh r_k - e^{i\theta_k}\sinh r_k}\right)$$

with $k = 1, 2$.

The time evolution of the wave packet is then calculated by substituting equations (16) and (18) into equation (4). This leads to

$$\begin{aligned} \psi(\mathbf{x}, \mathbf{y}, \Delta t) = &\left(\frac{1}{\pi\hbar}\right)^{1/2} \exp\left(\frac{-i\Delta t}{\hbar}\hat{V}_\lambda\right) \\ &\times \exp\left(\frac{-i\Delta t}{\hbar}\hat{H}_0\right)\psi(\mathbf{x}, \mathbf{y}, \mathbf{0}). \end{aligned} \tag{19}$$

In order to calculate $\psi(\mathbf{x}, \mathbf{y}, \Delta t)$, we make the assumption that the action of the unitary operator involving $\hat{H}_0$ on $\psi(\mathbf{x}, \mathbf{y}, \mathbf{0})$ gives only a phase factor. This is an approximation since while the vacuum state is an eigenfunction of $\hat{H}_0$, the squeezed vacuum state is not. However, if we consider a slightly squeezed vacuum state with r_k small, the error made in the approximation is small, which will be duly verified through results determined from numerical computation. Hence, we have the following expression for the short time evolved wave packet:

$$\psi(\mathbf{x},\mathbf{y},\Delta t) \approx \left(\frac{1}{\pi\hbar}\right)^{1/2} N_1 N_2 \exp\left(-\frac{i}{\hbar}\Phi\Delta t\right) \times \exp\left(\frac{-i\Delta t}{\hbar}\hat{V}_\lambda\right) \times \exp\left(\frac{-\left(\mathbf{x}^2/\eta_1+\mathbf{y}^2/\eta_2\right)}{2\hbar}\right), \quad (20)$$

where $\exp\left(-i\Phi\Delta t/\hbar\right)$ is the phase factor resulting from the above assumption. Note that we will ignore this phase factor in our subsequent calculation since it has no bearing on the results of linear entropy.

5 Linear entropy under the general interaction potential

Next, let us substitute the general expression of the interaction term $\hat{V}_\lambda = \lambda\mathbf{x}^m\mathbf{y}^n$ into equation (20). We get

$$\psi(\mathbf{x},\mathbf{y},\Delta t) = \left(\frac{1}{\pi\hbar}\right)^{1/2} N_1 N_2 \exp\left(\frac{-i\Delta t}{\hbar}\lambda\mathbf{x}^m\mathbf{y}^n\right) \times \exp\left(\frac{-(\mathbf{x}^2/\eta_1+\mathbf{y}^2/\eta_2)}{2\hbar}\right).$$

The reduced density function of subsystem (1) is then given by:

$$\rho_1(\mathbf{x},\mathbf{z}) = \left(\frac{1}{\pi\hbar}\right)|N_1|^2|N_2|^2 \exp\left(\frac{-\left(\mathbf{x}^2/\eta_1+\mathbf{z}^2/\eta_1^*\right)}{2\hbar}\right) \times \int_{-\infty}^{\infty} \exp\left(\frac{-\mathbf{y}^2}{2\hbar}\left(1/\eta_2+1/\eta_2^*\right)\right) \times \exp\left(\frac{-i\Delta t\lambda}{\hbar}\left(\mathbf{x}^m-\mathbf{z}^m\right)\mathbf{y}^n\right)dy.$$

Expanding the exponential term containing $\mathbf{y}^n$, we obtain

$$\rho_1(\mathbf{x},\mathbf{z}) = \left(\frac{1}{\pi\hbar}\right)|N_1|^2|N_2|^2 \exp\left(\frac{-\left(\mathbf{x}^2/\eta_1+\mathbf{z}^2/\eta_1^*\right)}{2\hbar}\right) \times \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{-i\lambda\Delta t}{\hbar}\right)^k\left(\mathbf{x}^m-\mathbf{z}^m\right)^k \times \int_{-\infty}^{\infty}\exp\left(\frac{-\Re(\eta_2)\mathbf{y}^2}{\hbar|\eta_2|^2}\right)\mathbf{y}^{nk}dy.$$

The y integral gives a non-zero contribution only when nk is an even number or zero. Here, $\Re(\eta_2)$ denotes the real part of the variable η_2. By evaluating the standard Gaussian integral, we obtain the reduced density function of subsystem (1) which is associated with the variable $\mathbf{x}$ as follows:

$$\rho_1(\mathbf{x},\mathbf{z}) = \left(\frac{1}{\pi\hbar}\right)|N_1|^2|N_2|^2 \times \exp\left(\frac{-\left(\mathbf{x}^2/\eta_1+\mathbf{z}^2/\eta_1^*\right)}{2\hbar}\right) \times \sum_{k=0}^{\infty}\frac{1}{k!}-\left(\frac{-i\lambda\Delta t}{\hbar}\right)^k\left(\mathbf{x}^m-\mathbf{z}^m\right)^k \times \hbar^{\frac{(nk+1)}{2}}\left(\frac{|\eta_2|^2}{\Re(\eta_2)}\right)^{\frac{(nk+1)}{2}}\Gamma\left(\frac{nk+1}{2}\right).$$

It has already been mentioned that the linear entropy $\delta(t)$ of the reduced density function $\rho_1(\mathbf{x},\mathbf{z})$ can also be defined as a double integral (see Eq. (9)). This definition is best suited for analytical calculation of the linear entropy of entanglement. Thus, to find the linear entropy, it is necessary to first find the trace of the square of $\rho_1(\mathbf{x},\mathbf{z})$:

$$\mathrm{Tr}(\rho_1^2(\mathbf{x},\mathbf{z})) = \left(\frac{1}{\pi\hbar}\right)^2|N_1|^4|N_2|^4 \times \sum_{k,k'=0}^{\infty}(-1)^{k'}\frac{1}{k!k'!}\left(\frac{-i\lambda\Delta t}{\hbar}\right)^{k+k'} \times \hbar^{\frac{nk+1+nk'+1}{2}}\left(\frac{|\eta_2|^2}{\Re(\eta_2)}\right)^{\frac{(nk+1+nk'+1)}{2}} \times \Gamma\left(\frac{nk+1}{2}\right)\Gamma\left(\frac{nk'+1}{2}\right) \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{(\mathbf{x}^2+\mathbf{z}^2)}{\hbar}\frac{\Re(\eta_1)}{|\eta_1|^2}\right) \times \left(\mathbf{x}^m-\mathbf{z}^m\right)^{k+k'}dxdz. \quad (21)$$

In order to evaluate the double integral in equation (21), we are required to perform a Cartesian to polar-coordinate transformation. This leads to the following result:

$$\mathrm{Tr}\left(\rho_1^2(\mathbf{x},\mathbf{z})\right) = \left(\frac{1}{\pi\hbar}\right)^2|N_1|^4|N_2|^4 \times \sum_{k,k'=0}^{\infty}(-1)^{k'}\frac{1}{k!k'!}\left(\frac{-i\lambda\Delta t}{\hbar}\right)^{k+k'} \times \hbar^{\frac{nk+1+nk'+1}{2}}\left(\frac{|\eta_2|^2}{\Re(\eta_2)}\right)^{\frac{nk+1+nk'+1}{2}} \times \Gamma\left(\frac{nk+1}{2}\right)\Gamma\left(\frac{nk'+1}{2}\right) \times \int_0^{\infty}\exp\left(\frac{-r^2\Re(\eta_1)}{\hbar|\eta_1|^2}\right)r^{m(k+k')+1}dr \times \int_0^{2\pi}\left(\cos^m\phi-\sin^m\phi\right)^{k+k'}d\phi. \quad (22)$$

After solving the Gaussian integral, we obtain

$$\begin{aligned}\mathrm{Tr}\left(\rho_1^2(\mathbf{x},\mathbf{z})\right) &= \left(\frac{1}{\pi}\right)^2 |N_1|^4|N_2|^4\left(\frac{|\eta_1|^2|\eta_2|^2}{\Re(\eta_1)\Re(\eta_2)}\right)\\ &\times\sum_{k,k'=0}^{\infty}\frac{(-1)^{k'}}{k!k'!}(-i\lambda\Delta t)^{k+k'}\\ &\times\frac{1}{2}\hbar^{((n+m)/2-1)(k+k'))}\\ &\times\left(\frac{|\eta_1|^2}{\Re(\eta_1)}\right)^{\frac{m(k+k')}{2}}\left(\frac{|\eta_2|^2}{\Re(\eta_2)}\right)^{\frac{n(k+k')}{2}}\\ &\times\Gamma\left(\frac{nk+1}{2}\right)\Gamma\left(\frac{nk'+1}{2}\right)\\ &\times\Gamma\left(\frac{m(k+k')}{2}+1\right)I_{m,k+k'},\end{aligned}\tag{23}$$

where $I_{m,k+k'}$ is a ϕ integral given by:

$$I_{m,k+k'} = \int_0^{2\pi}\left(\cos^m\phi - \sin^m\phi\right)^{k+k'}d\phi.\tag{24}$$

Since $I_{m,k+k'}$ is independent of $\hbar$ and also independent of the variables η_1 and η_2, it is not necessary to evaluate this integral to find the $\hbar$-dependence of the short time entanglement. Furthermore, by using basic trigonometric and hyperbolic identities, it can be easily shown that

$$|N_1|^4|N_2|^4\left(\frac{|\eta_1|^2|\eta_2|^2}{\Re(\eta_1)\Re(\eta_2)}\right) = 1.\tag{25}$$

The linear entropy can be evaluated by substituting equation (23) into equation (9), and we can clearly see that the first term in the summation, i.e., $k = k' = 0$, gives unity and it will cancel with the +1 term in the definition of $\delta(t)$. Hence, we have the following expression for the linear entropy:

$$\begin{aligned}\delta(t) &= \left(\frac{1}{\pi}\right)^2\sum_{k,k'=0,(k=k'\neq 0)}^{\infty} i^{(k+k')}\frac{(-1)^{(k+1)}}{k!k'!}\\ &\times(\lambda\Delta t)^{k+k'}\times\frac{1}{2}\hbar^{((n+m)/2-1)(k+k'))}\\ &\times\left(\frac{|\eta_1|^2}{\Re(\eta_1)}\right)^{\frac{m(k+k')}{2}}\left(\frac{|\eta_2|^2}{\Re(\eta_2)}\right)^{\frac{n(k+k')}{2}}\\ &\times\Gamma\left(\frac{nk+1}{2}\right)\Gamma\left(\frac{nk'+1}{2}\right)\Gamma\left(\frac{m(k+k')}{2}+1\right)\\ &\times I_{m,k+k'}.\end{aligned}\tag{26}$$

Now, we have to determine the lowest power of $\hbar$ within $\delta(t)$. We have already known that to get a nonzero contribution, nk should be an even number. Similarly, for the second summation index k', to get a nonzero term, nk' should also be an even number. Hence, the constraints to yield nonzero values of the integrals imply that nk' and nk should be even integers or zero. Moreover, based on the real positive definiteness property of the linear entropy and using the sign of $i^{(k+k')}\times(-1)^{(k+1)}$, we realize the conditions: $k + k'$ must be even, and $k + k' \geq 2$. After considering all the possible options for the smallest values of k and k' depending on the nature of m and n, we come to the conclusion that the k and k' which obey $k+k' = 2$ shall give the non-vanishing lowest order terms for $\hbar$. This tells us that the lowest order exponent for the time interval in the short time semiclassical regime is two, i.e., $k + k' = 2$. At the same time, this implies that the minimum power of the $\hbar$ term in the expression for $\delta(t)$ is given by:

$$\text{minimum power of } \hbar = m + n - 2.\tag{27}$$

Hence, the general short time entanglement for the squeezed vacuum state under the interaction potential $\lambda\mathbf{x}^m\mathbf{y}^n$ takes the form

$$\delta(t) = \kappa\Delta t^2\lambda^2\hbar^{m+n-2}\left(\frac{|\eta_1|^2}{\Re(\eta_1)}\right)^m\left(\frac{|\eta_2|^2}{\Re(\eta_2)}\right)^n,\tag{28}$$

where the coefficient κ is given by

$$\begin{aligned}\kappa = &\sum_{k,k',(k+k'=2)}\frac{i^{(k+k')}}{2\pi^2}\frac{(-1)^{k+1}}{k!k'!}\\ &\times\Gamma\left(\frac{nk+1}{2}\right)\Gamma\left(\frac{nk'+1}{2}\right)\Gamma\left(\frac{m(k+k')}{2}+1\right)\\ &\times I_{m,k+k'}.\end{aligned}\tag{29}$$

Then, by taking $k' = k - 2$, we can replace the double summation with a single dummy summation with index j such that

$$\kappa = \sum_j \kappa_j.\tag{30}$$

If we were to consider the special case $\theta_1 = \theta_2 = 0$, the squeezing parameters are real and consequently η_1 and η_2 are also real. Hence, for real squeezing parameters, our expression of the linear entropy reduces to

$$\delta(t) = \kappa t^2\lambda^2\hbar^{m+n-2}|\eta_1|^m|\eta_2|^n.\tag{31}$$

In the case of coherent vacuum state, we have $|\eta_1| = 1$ and $|\eta_2| = 1$, and equation (31) of the linear entropy reduces to

$$\delta(t) = \kappa t^2\lambda^2\hbar^{m+n-2}.\tag{32}$$

From these results, it can be clearly seen that the $\hbar$ dependence of short time entanglement entropy depends on the nature of the interaction potential. More specifically, if the degree m and n have orders higher than unity, an $\hbar$ dependent short time entanglement dynamics is to be expected.

The type of interaction potentials that we have studied appears in many models, especially the interaction potential $\lambda\mathbf{xy}$ occurs in the study on the entanglement dynamics of the coupled Kerr oscillator [1]. Instead of exploring this system, we shall investigate into the coupled harmonic oscillators with the interaction term $\lambda\mathbf{xy}$. Furthermore, our general result on the linear entropy can be easily applied to diverse coupled oscillator systems such as the Barbanis-Contopulos, the Hénon-Heiles and the Pullen-Edmonds.

6 Application of analytical results to coupled oscillator systems

6.1 Squeezed vacuum under the linearly coupled harmonic oscillator

Let us first consider the following Hamiltonian with a linear interaction term in x and y:

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{2}\left(x^2 + y^2\right) + \lambda xy. \tag{33}$$

Applying our earlier theoretical analysis, we found that the value of the coefficient $\kappa = \frac{1}{2}$. Hence, the expression for the linear entropy given by equation (28) becomes

$$\delta(t) = \frac{1}{2}\Delta t^2\lambda^2\left(\frac{|\eta_1|^2}{\Re(\eta_1)}\right)\left(\frac{|\eta_2|^2}{\Re(\eta_2)}\right). \tag{34}$$

For real squeezing parameter, this expression reduces to

$$\delta(t) = \frac{1}{2}\Delta t^2\lambda^2|\eta_1||\eta_2|. \tag{35}$$

For coherent vacuum state, $|\eta_1| = 1$ and $|\eta_2| = 1$. The linear entropy of equation (35) then further simplifies to

$$\delta(t) = \frac{1}{2}\Delta t^2\lambda^2. \tag{36}$$

This result clearly shows that for interaction potential $\hat{V_\lambda} = \lambda xy$, the short time linear entropy of entanglement is independent of the Planck constant $\hbar$.

6.2 Squeezed vacuum under the Barbanis-Contopoulos Hamiltonian

This Hamiltonian was introduced by the astronomers Contopoulos and Barbanis [20,21]. It is written in the following form:

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{2}\left(x^2 + y^2\right) + \lambda x^2 y. \tag{37}$$

The linear entropy for this system can be obtained by employing equation (28) directly with $m = 2$ and $n = 1$. This leads to the short time linear entropy:

$$\delta(t) = \frac{1}{2}\Delta t^2\lambda^2\hbar\left(\frac{|\eta_1|^2}{\Re(\eta_1)}\right)^2\left(\frac{|\eta_2|^2}{\Re(\eta_2)}\right). \tag{38}$$

For real squeezing parameters, i.e., $\theta_1 = \theta_2 = 0$, the linear entropy reduces to

$$\delta(t) = \frac{1}{2}\Delta t^2\lambda^2\hbar|\eta_1|^2|\eta_2|. \tag{39}$$

For the coherent vacuum state $|\eta_1| = 1$ and $|\eta_2| = 1$, the linear entropy in equation (39) becomes

$$\delta(t) = \frac{1}{2}\Delta t^2\lambda^2\hbar. \tag{40}$$

6.3 Squeezed vacuum under the Hénon-Heiles Hamiltonian

The Hénon-Heiles system was first studied by the astronomers Hénon and Heiles in 1964 [22] in the context of analyzing the constants of motion in galactic dynamics. Due to its simplicity and rich dynamical properties, chaos theorists had extensively explored different classical dynamical aspects of this system [23–27]. The Hénon-Heiles Hamiltonian can be written as

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{2}\left(x^2 + y^2\right) + \lambda\left(x^2 y - \frac{y^3}{3}\right). \tag{41}$$

Unlike the earlier Hamiltonian, the Hénon-Heiles system's interaction potential $\hat{V_\lambda} = \lambda(\mathbf{x}^2\mathbf{y} - \mathbf{y}^3/3)$ possesses an extra term $\mathbf{y}^3/3$. By means of the same theoretical arguments as given in Section 4, we arrive at the reduced density function of the first subsystem ρ_1 as follows:

$$\rho_1(\mathbf{x},\mathbf{z}) = \left(\frac{1}{\pi\hbar}\right)|N_1|^2|N_2|^2\exp\left(\frac{-(\mathbf{x}^2/\eta_1 + \mathbf{z}^2/\eta_1{}^*)}{2\hbar}\right) \times \int_{-\infty}^{\infty}\exp\left(\frac{-\mathbf{y}^2\Re(\eta_2)}{\hbar|\eta_2|^2} - \frac{i\lambda\Delta t}{\hbar}(\mathbf{x}^2 - \mathbf{z}^2)\mathbf{y}\right)dy. \tag{42}$$

Here, we notice that the $\mathbf{y}^3$ term vanishes in the reduced density function and it does not contribute to the entanglement in the short time regime. Hence, we can directly apply our general expression given by equation (28) to obtain the linear entropy:

$$\delta(t) = \frac{1}{2}\Delta t^2\lambda^2\hbar\left(\frac{|\eta_1|^2}{\Re(\eta_1)}\right)^2\left(\frac{|\eta_2|^2}{\Re(\eta_2)}\right). \tag{43}$$

Again, for real squeezing parameters, the linear entropy reduces to

$$\delta(t) = \frac{1}{2}\Delta t^2\lambda^2\hbar|\eta_1|^2|\eta_2|. \tag{44}$$

For the coherent vacuum state, the linear entropy in equation (44) becomes

$$\delta(t) = \frac{1}{2}\Delta t^2\lambda^2\hbar. \tag{45}$$

Here, the short time linear entropy is identical to the Barbanis-Contopoulos Hamiltonian since both systems share the same interaction potential $\mathbf{x}^2\mathbf{y}$.

Table 1.

Hamiltonian	Squeezed Vaccum	Coherent Vaccum ($\theta = 0$)
Linearly coupled harmonic oscillator	$\delta(t) = \frac{1}{2}\Delta t^2\lambda^2\left(\frac{\lvert\eta_1\rvert^2}{\Re(\eta_1)}\right)\left(\frac{\lvert\eta_2\rvert^2}{\Re(\eta_2)}\right)$	$\delta(t) = \frac{1}{2}\Delta t^2\lambda^2$
Barbanis-Contopoulos	$\delta(t) = \frac{1}{2}\hbar\Delta t^2\lambda^2\left(\frac{\lvert\eta_1\rvert^2}{\Re(\eta_1)}\right)^2\left(\frac{\lvert\eta_2\rvert^2}{\Re(\eta_2)}\right)$	$\delta(t) = \frac{1}{2}\hbar\Delta t^2\lambda^2$
Hénon-Heiles	$\delta(t) = \frac{1}{2}\hbar\Delta t^2\lambda^2\left(\frac{\lvert\eta_1\rvert^2}{\Re(\eta_1)}\right)^2\left(\frac{\lvert\eta_2\rvert^2}{\Re(\eta_2)}\right)$	$\delta(t) = \frac{1}{2}\hbar\Delta t^2\lambda^2$
Pullen-Edmonds	$\delta(t) = \frac{1}{2}\hbar^2\Delta t^2\lambda^2\left(\frac{\lvert\eta_1\rvert^2}{\Re(\eta_1)}\right)^2\left(\frac{\lvert\eta_2\rvert^2}{\Re(\eta_2)}\right)^2$	$\delta(t) = \frac{1}{2}\hbar^2\Delta t^2\lambda^2$

6.4 Squeezed vacuum under the Pullen-Edmonds Hamiltonian

The Pullen-Edmonds Hamiltonian can be written as:

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{2}\left(x^2 + y^2\right) + \lambda x^2 y^2. \tag{46}$$

Pullen and Edmonds had shown that the classical dynamical behaviour of this model can range from purely regular, to a mixture of regular and chaos, and to fully chaotic [29]. This Hamiltonian had been used extensively in many works related to classical and quantum dynamics [30–32,43,44]. By employing our general analytical expression given by equation (28) with $m = 2$ and $n = 2$, the short time linear entropy of entanglement for the squeezed vacuum state in the Pullen-Edmonds Hamiltonian is given by

$$\delta(t) = \frac{1}{2}\Delta t^2\lambda^2\hbar^2\left(\frac{|\eta_1|^2}{\Re(\eta_1)}\right)^2\left(\frac{|\eta_2|^2}{\Re(\eta_2)}\right)^2. \tag{47}$$

For the squeezed vacuum state with zero squeezing angles, this expression becomes

$$\delta(t) = \frac{1}{2}\Delta t^2\lambda^2\hbar^2|\eta_1|^2|\eta_2|^2. \tag{48}$$

Also for the coherent vacuum state $|\eta_1| = 1$ and $|\eta_2| = 1$, the linear entropy reduces to

$$\delta(t) = \frac{1}{2}\Delta t^2\lambda^2\hbar^2. \tag{49}$$

Finally, we summarize our results for all the above systems in Table 1 for the sake of comparison. From Table 1, we clearly see that the entanglement dynamics is independent of the Planck constant for the linearly coupled oscillator. For all the generic Hamiltonians which possess chaotic behavior, there is an $\hbar$ dependent entanglement dynamics according to the degree of their interaction term. It is important to mention that these theoretical results are valid only in the short time regime and the squeezing on the vacuum state is assumed to be small.

7 Theoretical and numerical comparison of the short time linear entropy

In this section, we detail our results from numerical computations and compare them against our analytical results. First, we study the entanglement dynamics of the coherent vacuum state for the linearly coupled harmonic oscillators, the Barbanis-Contopoulos, the Hénon-Heiles and the Pullen-Edmonds Hamiltonian, respectively. Then, we focus on the entanglement of the squeezed vacuum state, and finally we explore the $\hbar$ dependence of the entanglement for these systems.

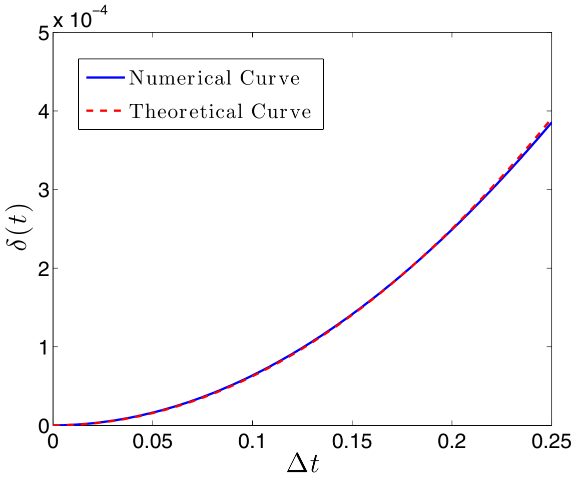
7.1 Coherent vacuum state and the linear entropy

It is worthwhile to first observe the validity of our theoretical results against the numerical results for the case of vacuum state. For this purpose, Figure 1 plots the linear entropy against time for the four systems of interest. Note that the solid blue curve and the dashed red curve show the entanglement entropy evaluated numerically and theoretically, respectively. The results in Figure 1 clearly demonstrate the close agreement between the analytical and numerical results in the short time regime. In fact, a similar Δt^2 dependence of the entanglement has been observed in previous works [1,2] and a similar behavior has also been found for the idempotency defect in a dissipative system [45].

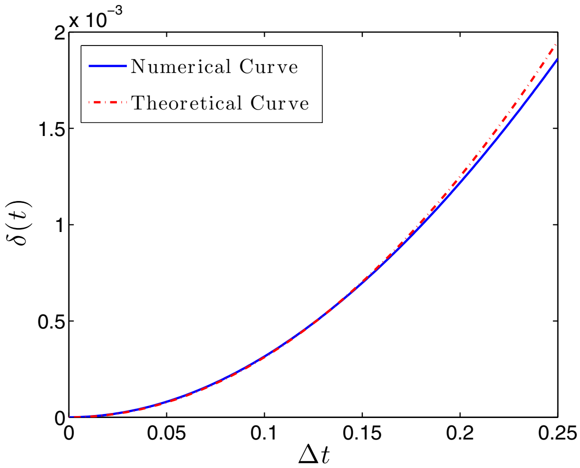
It can be observed from Figure 1 that the difference between the theoretical and numerical results increases with time. This is due to the theoretical assumption that the time evolution operator containing $\hat{H}_0$ gives only a phase factor, with the error from this assumption increasing with time. In addition, the truncation of the Zassenhaus formula and $\delta(t)$ given by equation (28) where higher order terms are dropped also introduces error that increases with time. This explains the observed difference between the theoretical and the numerical results in the long time regime.

7.2 Squeezed vacuum state and the linear entropy

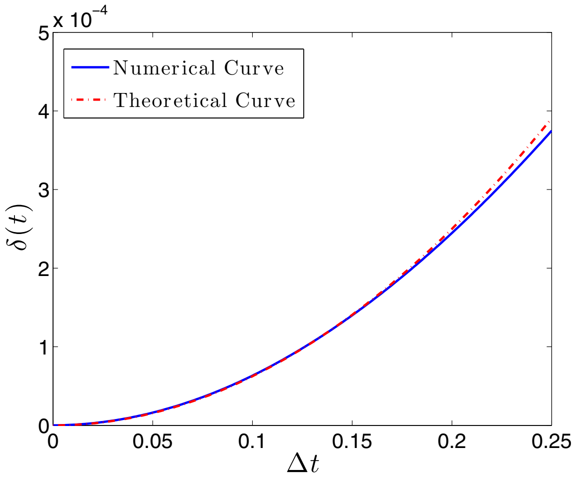
In Figure 2, the numerical and theoretical linear entropy of entanglement for the squeezed vacuum state is shown



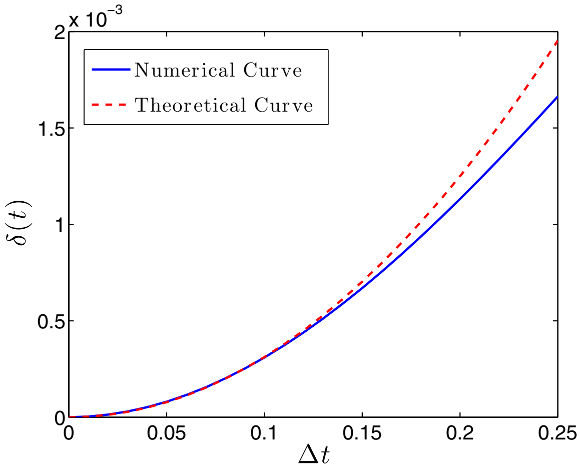
(a) Linearly coupled harmonic oscillator with $\lambda = \frac{1}{\sqrt{80}}$



(b) Barbanis-Contopoulos Hamiltonian with $\lambda = 0.25$



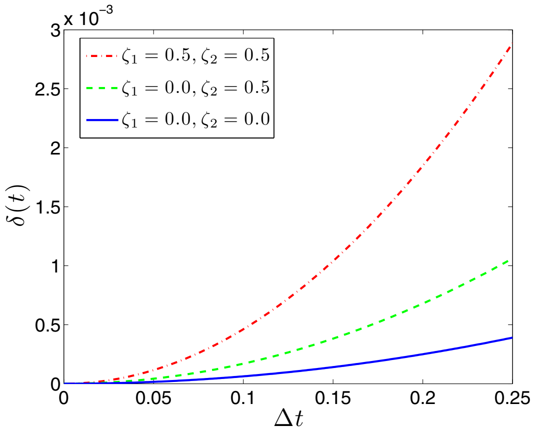
(c) Hénon-Heiles Hamiltonian with $\lambda = \frac{1}{\sqrt{80}}$



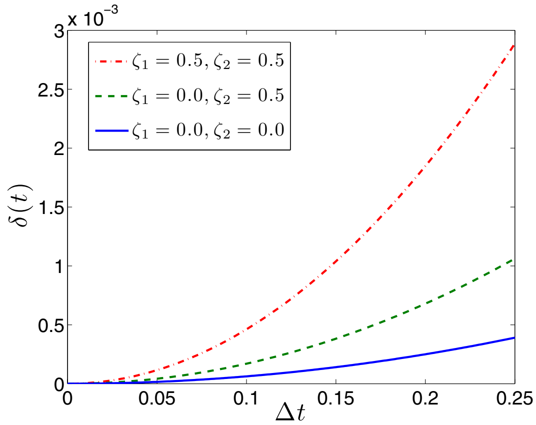
(d) Pullen-Edmonds Hamiltonian with $\lambda = 0.25$

Fig. 1. Plots of the linear entropy of entanglement $\delta(t)$ versus short time Δt for initial coherent vacuum state. The solid blue curve and the dashed red curve show the entanglement entropy evaluated numerically and analytically, respectively. The plots show that both the analytical and numerical results are in good agreement within the short time regime. Note that we have employed the following parameter values: $\zeta_1 = \zeta_2 = 0$, and $\hbar = 1$.

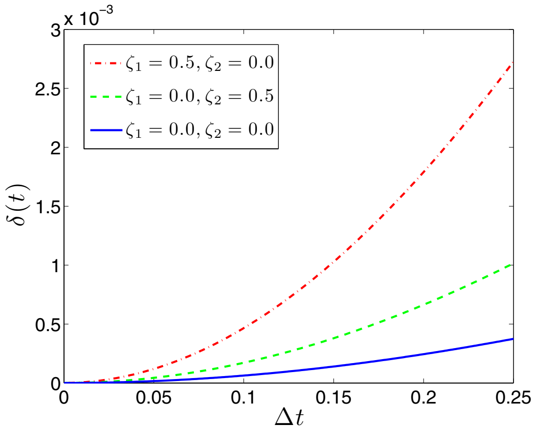
for three different Hamiltonians. It is observed that these results of linear entropy of entanglement for the squeezed vacuum state is in good agreement with the theoretical prediction in the short time regime. It has already been noted that squeezing can enhance entanglement and our result clearly demonstrates this phenomenon. In fact, equation (28) clearly indicates that the linear entropy depends on the values of η_1 and η_2 as well as the exponents of the interaction terms, i.e., m and n. As the values of m and n become larger, we expect the entanglement growth via squeezing to become larger. Moreover, it can be ascertained that the entanglement growth is higher when both modes of the wave function are simultaneously squeezed, i.e., ζ_1 and ζ_2 are nonzero. It is apparent that initial squeezed state always leads to higher entanglement compared to initial coherent vacuum state within the short time regime. For the Hénon-Heiles Hamiltonian, there occurs the interesting situation where the exponents of η_1 and η_2 as shown in equation (44) are not symmetric. This asymmetry in the exponent of η_1 and η_2 comes from the asymmetry of the Hénon-Heiles interaction term $\mathbf{x}^2\mathbf{y}$. This asymmetry can bring about a unique property: swapping of the squeezing parameter ζ_1 and ζ_2 of the initial squeezed wave function can give rise to a different entanglement. Specifically, if one aims to attain a higher rate of entanglement growth for this case, it would be better to squeeze the field modes associated with the $\mathbf{x}$ variable than the $\mathbf{y}$ variable. This fact can be easily established from Figures 2c and 2d. Indeed, these figures clearly show that the red curve, where the $\mathbf{x}$ variable is squeezed ($\zeta_1 = 0.5, \zeta_2 = 0.0$), has a higher entanglement compared to the green curve where the $\mathbf{y}$ variable is squeezed ($\zeta_1 = 0.0, \zeta_2 = 0.5$). Both the numerical and theoretical results illustrate the same outcome. In the case of



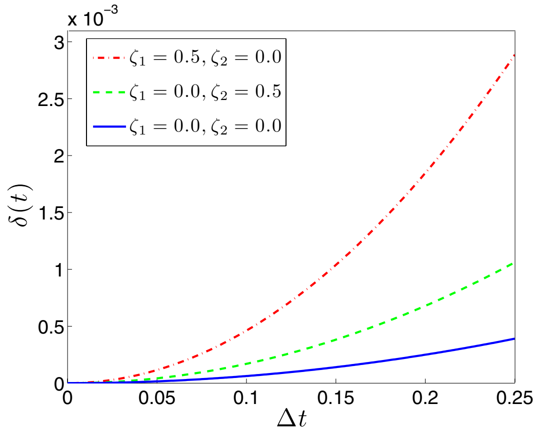
(a) Numerical result: linearly coupled harmonic oscillator $\lambda = \frac{1}{\sqrt{80}}$



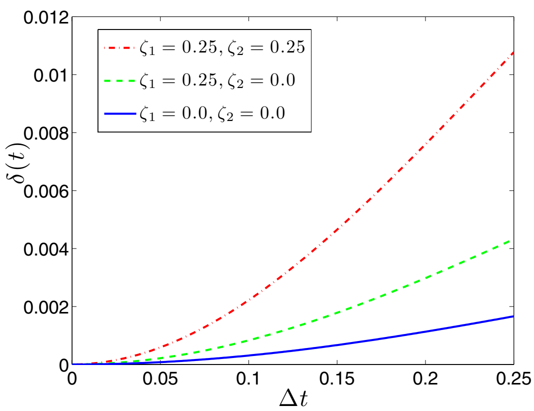
(b) Theoretical result: linearly coupled harmonic oscillator $\lambda = \frac{1}{\sqrt{80}}$



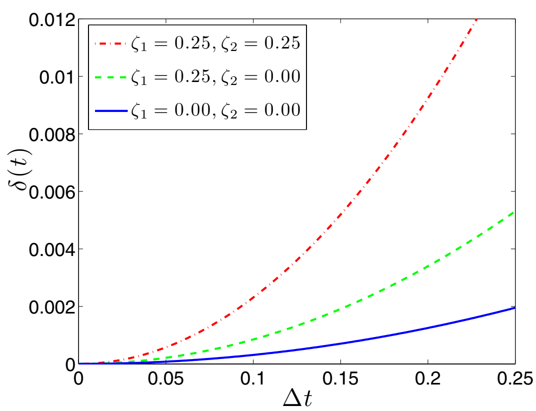
(c) Numerical result: Hénon-Heiles Hamiltonian $\lambda = \frac{1}{\sqrt{80}}$



(d) Theoretical result: Hénon-Heiles Hamiltonian $\lambda = \frac{1}{\sqrt{80}}$

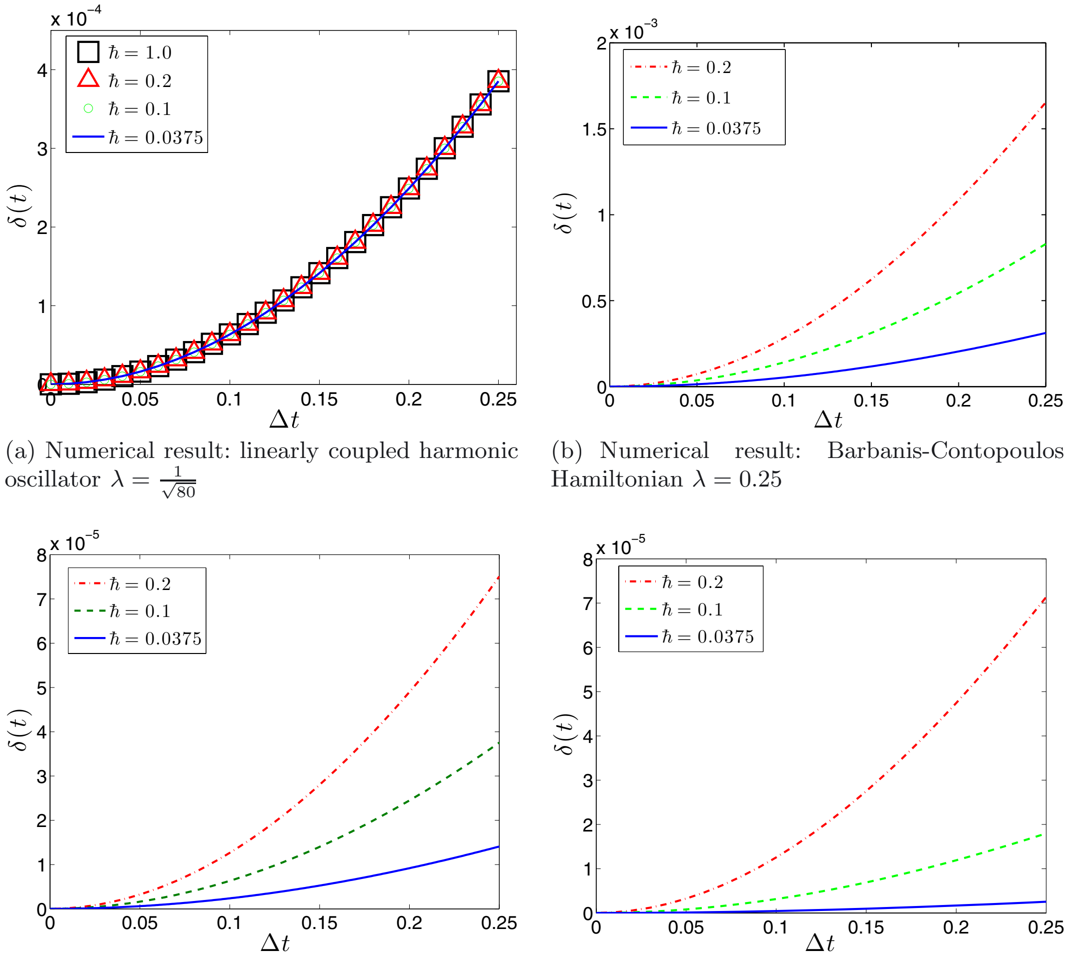


(e) Numerical result: Pullen-Edmonds Hamiltonian $\lambda = 0.25$



(f) Theoretical result: Pullen-Edmonds Hamiltonian $\lambda = 0.25$

Fig. 2. Plots of the entanglement entropy $\delta(t)$ versus short time Δt, where curves of different colors indicate initial vacuum states with different squeezing parameters ζ_1 and ζ_2. The figures clearly show that initial squeezing enhances the entanglement entropy for all the three systems. Note that similar results are found for the Barbanis-Contopoulos Hamiltonian although they are not illustrated. We have employed $\hbar = 1$ in all the plots in this figure.



(a) Numerical result: linearly coupled harmonic oscillator $\lambda = \frac{1}{\sqrt{80}}$

(b) Numerical result: Barbanis-Contopoulos Hamiltonian $\lambda = 0.25$

(c) Numerical result: Hénon-Heiles Hamiltonian $\lambda = \frac{1}{\sqrt{80}}$

(d) Numerical result: Pullen-Edmonds Hamiltonian $\lambda = 0.25$

Fig. 3. Plots of the entanglement entropy $\delta(t)$ versus the short time Δt for different values of the Planck constant $\hbar$. It can be observed from these figures that as $\hbar \to 0$, the linear entropy of entanglement tends to zero for the Barbanis-Contopoulos, the Hénon-Heiles and the Pullen-Edmonds Hamiltonian. However, for the linearly coupled harmonic oscillator, $\delta(t)$ is observed to be independent of $\hbar$. This can be discerned through the different curves overlapping on each other to form a single line such that different markers have to be used to distinguish between them. Note that these results are consistent with our theoretical prediction. We have employed $\zeta_1 = \zeta_2 = 0.0$ for all the plots in this figure.

symmetric interaction, such a swapping of squeezing parameter would not cause any changes to the entanglement dynamics, which have been verified by us both analytically and numerically.

7.3 Dependence of linear entropy on the Planck constant

Theoretically, we have observed that the short time entanglement is directly proportional to the Planck constant for the Barbanis-Contopoulos and the Hénon-Heiles Hamiltonian. As for the Pullen-Edmonds Hamiltonian, we obtain instead a $\hbar^2$ dependence. On the other hand, the short time linear entropy is found to be independent of $\hbar$ for the linearly coupled harmonic oscillator. In Figure 3, we have plotted the dynamics of linear entropy initiated by the coherent vacuum state for different values of the Planck constant $\hbar$. These plots were obtained numerically for each of the four systems. From Figures 3b–3d, we observe that as $\hbar \to 0$, the linear entropy of entanglement tends to zero in accordance with our theoretical prediction. On the contrary, the linearly coupled harmonic oscillator displays an $\hbar$ independence of the linear entropy as indicated by our theoretical analysis. Thus, these results affirm the fact that the linear entropy should tend to zero as $\hbar \to 0$ if the interaction potential has a sum of degree that is higher than two, since theoretically the power of $\hbar$ is given by $m + n - 2$.

It is important to note that in our final result the linear entropy always depends on the square of the time interval Δt at the short time regime in accordance with previous works [1,2], except for the $\hbar$ dependence. However, we would like to emphasize that our results for the

$\hbar$ dependence of the linear entropy do not contradict with this previous general results. In fact, the $\hbar$ independent entanglement entropy that we obtain for the linearly coupled harmonic oscillator is identical to that of reference [1]. The expression of the semiclassical linear entropy consists of a product of the square of the short time interval and the summation of $\hbar$ terms of different order. More precisely, there exist nonzero values for the $\hbar$ terms with nonnegative exponents, and in the semiclassical limit, the zeroth order term of $\hbar$ dominates. For certain potentials, like the potentials that we have investigated, the zeroth order term of $\hbar$ can vanish. In consequence, a higher order term of $\hbar$ gives the dominant contribution, resulting in an $\hbar$ dependent entanglement dynamics.

8 Conclusion

We have analyzed the short time quantum entanglement of the squeezed vacuum for a general interaction potential of the form $V_\lambda = \lambda \mathbf{x}^m \mathbf{y}^n$. We have applied our general results to the specific cases of linearly coupled harmonic oscillator, the Barbanis-Contopoulos, the Hénon-Heiles and the Pullen-Edmonds Hamiltonian, where we have explored both the initial coherent and squeezed coherent vacuum state. We have found good correspondence between the analytical and numerical results within the short time regime. More significantly, we have uncovered that initial squeezing invariably enhances the entanglement in this regime. It is interesting that for the Hénon-Heiles as well as the Barbanis-Contopoulos Hamiltonian, the linear entropy of entanglement has an asymmetric dependence on the squeezing parameters for the $\mathbf{x}$ and $\mathbf{y}$ modes. Hence, a swapping of the squeezing parameter of the initial wave function can give rise to a different entanglement dynamics. For the linearly coupled harmonic oscillator, the entanglement dynamics is found to be independent of the Planck constant $\hbar$. Conversely, for the Barbanis-Contopoulos, the Hénon-Heiles and the Pullen-Edmonds Hamiltonian, the short time entanglement entropy tends to zero in the semiclassical limit where the Planck constant goes to zero. As a general result, we have found that the short time linear entropy of entanglement depends on $\hbar$ raised to an exponent that is the sum of the degree of the interaction term minus two.

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