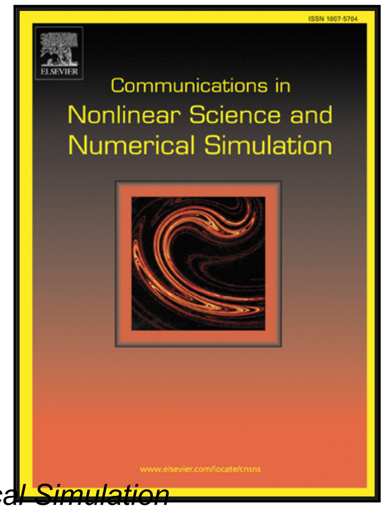

Supply based on demand dynamical model

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Highlights

- A supply model in uncertainty of demand forecast is proposed.
- The model allows to study several supplier strategies.
- It is shown that small changes in price elasticity lead to very different dynamics.
- The model shows chaotic behavior for a wide range of parameters.
- The proposed model is able to reproduce market collapses.

Supply based on demand dynamical model

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Abstract

We propose and numerically analyze a simple dynamical model that describes the firm behaviors under uncertainty of demand. Iterating this simple model and varying some parameter values, we observe a wide variety of market dynamics such as equilibria, periodic, and chaotic behaviors. Interestingly, the model is also able to reproduce market collapses.

1. Introduction

Many firms need to decide the number of units of a certain product to produce before they know how many products the market will demand in the next sales season. This problem is known as *uncertainty of demand forecast* and it has been widely studied in economics and supply chain management [1]. Being successful in predicting the future demand, might be crucial for the survival of any firm in a constantly changing market environment [2]. Many models involving demand uncertainty have been proposed in the literature. For example, in economics, monopoly pricing models under uncertainty of demand, considering the demand as a stochastic function [3, 4, 5], became popular after the consolidation of the idea of information asymmetry in the markets proposed by George A. Akerlof [6] in 1970. Another famous problem in management known as the *Newsvendor problem* traced back to the economist Edgeworth (1888), addresses the same question, focusing on optimizing the stocking process maximizing profits, using some data analysis tools and statistical techniques [7]. In supply chain

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management, statistical methods, like time series analysis or linear regression [8], are commonly used to estimate the future demand. There are deterministic methods as well to deal with demand uncertainty, for example deterministic demand inventory models, such as the economic order quantity (EOQ) model. Agent based modeling [9, 10] is a novel technique used by academics and industry researches to face demand uncertainty in the markets. This modeling approach is a part of a multidisciplinary way of thinking in economics [11, 12, 13] called complexity economics that has recently become very popular.

In this work, we will model a market where one firm is a price maker operating under uncertainty of demand. We focus on firms whose commercial activity involves producing or buying some stocks of a certain good with the purpose of selling them to obtain profits. These firms, mainly small, medium or entrepreneurs, do not spend too much resources in demand forecasting. They rely mainly on their buyers expectations among limited data sets of past sales, for example small stores, retailers or small manufacturers.

The model that we propose here, has been built in a similar way to a Dynamic Stochastic General Equilibrium (DSGE) model without the stochastic terms, focusing on the micro level of the economy. These kind of models have been adopted by policy makers all over the world to predict and even control the economy at the macro level [14, 15]. These models are built from three main blocks, where each one is a representation of some economic agent or a group of agents. The demand block represents the consumption of households, firms and even the government. The supply block represents the productive agents of the economy, and the policy block represents financial institutions like central banks [16]. These kind of models add to the general equilibrium models some simple dynamical interaction between the economical agents in addition to some stochastic external shocks.

The model presented here is highly inspired by the classic cobweb model [17] with the difference that, the firm decision of how many units to produce is sensitive to the quantity demanded instead of the market price. This model highlights two interesting dynamical aspects of the supplying process. First, as in the nonlinear version of the cobweb model, we present one possible dynamical procedure based on firms expectations [18, 19, 20, 21] that can lead to a chaotic dynamics. The second idea embodied in this work is that for a given quantity of supply the firm fixes some price that generates a demand feedback from the market. This information is needed to compute the quantity of supply, in the next time step. As in real markets, the firm reacts to these demand feedbacks, creating a rich price-quantity dynamics. We will show that in some cases the firm may push the market towards an equilibrium motivated by his selfish interests, selling all the stock. As Adam Smith once wrote: *"It is not from the benevolence of the butcher, the brewer, or the baker that we expect our dinner, but from their regard to their own self-interest..."* [22]. But in other cases the firm may produce irregular dynamics that may lead to a market collapse. We have found that the price elasticity of demand (PED) and the gross margin can play an important role in the stabilization of prices in the same way they can make the market crash.

The structure of the paper is as follows. Section 2 is devoted to the description of the supply based on demand model. Two types of firms and their behaviors are described in Section 3. In Section 4 we explore the dynamics of the model for several parameters. The global dynamics and results are described in section 5. In Section 6 we emphasize the idea that the final bifurcation means market collapse. We describe the influence of the price elasticity of demand (PED) on the global dynamics in Section 7. Finally, some conclusions are drawn in Section 8.

2. Description of the supply based on demand model

In our model, we consider a carefully chosen relation between demand, supply and price. Note that this relation is not the dynamical system that we will analyze. We will derive the dynamical system to study in Section 3, using the Eqs. 1-3, which shows how these variables are related,

$$D_{n+1} = a - bP_{n+1}, \tag{1}$$
$$S_{n+1} = D^{Exp}_{n+1}, \tag{2}$$
$$P_{n+1} = \tfrac{ATC}{1-M}, \tag{3}$$

where the quantities demanded and supplied, D_{n+1}, and S_{n+1}, and the price, P_{n+1} are assumed to be discrete functions of time. The parameters a and b are positive constants $a, b \geq 0$ and D^{Exp}_{n+1}, is the expected demand. The parameter M, is the gross margin added by the firm to obtain profits, where $0 \leq M < 1$. The variable ATC is the average total cost function of the product, which we will explain in detail later on.

The quantity demanded in the market depends mainly on the price of a given product. The price of the product in contrast, depends heavily on the average total cost function, which is directly linked to the quantity of supply. When the firm decides how many units is going to produce, it always estimates in some way the future demand, D^{Exp}_{n+1} [1]. The problem is, that the firm makes the decision of what quantity to supply, S_{n+1}, before it knows the reaction of the market to the price that

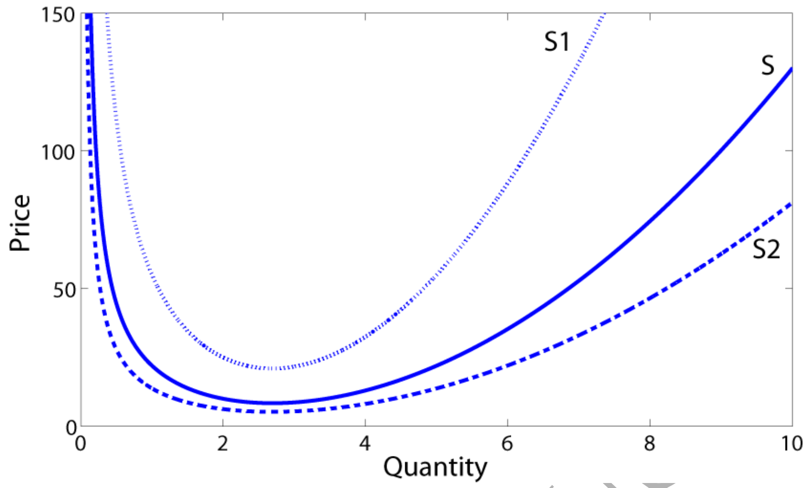

Figure 1: **The price-quantity function.** We have used the following function $P = \frac{1}{1-M} \cdot (\frac{F_c}{Q} + v - vQ + Q^2)$, to relate the price of the product with the quantity supplied, where P is the selling price of the product (cost + profits) and Q is the quantity of production. The average fix cost function is $\frac{F_c}{Q}$ where F_c is a positive constant and the average variable cost function is $v - vQ + Q^2$, where v is positive constant. The parameters are fixed as: $F_c = 10$ and $v = 4$. The supply curves S (solid line), $S1$ (dot line) and $S2$ (dash-dot line), correspond to the gross margin $M = 0.5$, $M = 0.8$, $M = 0.2$ respectively. When the firm increases, the gross margin M, the price increases and when the firm reduces the gross margin M, the price decreases.

it fixes. In this model, we assume that the firm does not know anything about the demand function. The only available information it has, is the quantity demanded at the price in which it sold its products in the last sales season. It is important to notice that the quantity demanded from the firm perspective is the sum of the total units sold and the stock rupture units (out of stock units). We assume an ordinary goods market in which, when the price increases, the consumption of the products decreases and vice versa. For simplicity, we assume a linear demand curve with negative slope as shown in Eq. 1. Before we proceed, we introduce two more mechanistic assumptions, that describe how the supplier operates in the market.

Assumption 1

The firm is the only one that sets and adjusts the price in light of circumstances.

In this model the firm is a price maker. Notice that after the firm launches the products into the market, no changes can be done in the quantity supplied, nor the price. The price structure is given by the ATC function and the gross margin, as shown in Eq. 3. Both building blocks are known and controlled by the firm.

After estimating the demand for the next period, the firm begins the production phase. It introduces its estimations in the ATC function to obtain its average total costs of production. We assume in this model that the average total costs is computed adding the fix costs to the variable cost per unit of good, divided by the total amount of goods produced. However, there are many possible ways to describe an ATC function. For instance, in many industries the price lists shown to the buyers are organized in a "piecewise function" fashion, where the price of the good is well established for every subset of quantities the buyer is willing to buy. But here, to stay faithful to the classical cost theory, we have chosen a typical continuous cubic total cost function, that gives rise to a quadratic ATC function that depends also on the quantity of production Q [23] as shown in Fig. 1. The average total cost function ATC of the firm will adopt a U-shape, when diminishing returns are present in the production process and the firm has variable costs. Applying this idea, when the firm increases the amount of production the average total cost of every unit of production decreases until it reaches some critical point from which every additional produced unit will increase the unit average total cost. In the decreasing side of the curve, the firm enjoys the economies of scale, that is, decreasing returns to scale. After crossing this point, every additional produced unit increments the average total cost of the firm, which implies diminishing returns to scale [24]. The U-shape of the ATC function as shown in Fig. 1 captures this idea. The quantity of production Q is the same as the quantity of supply, S_{n+1}, or the expected demand estimated by the firm earlier, as shown in Eqs. 2 and 4

$$ATC = \frac{F_c}{S_{n+1}} + v - vS_{n+1} + (S_{n+1})^2. \tag{4}$$

We assume that the variable cost, v, and the fix cost, F_c, are positive constants. The final step in this process is to add profits over the average total cost of the product, using the gross margin operator shown in Eq. 3. When M increases, the price function moves upwards, what leads to higher prices and when it decreases the price moves downwards what leads to cheaper products as shown in Fig. 1.

Assumption 2

The main goal of the firm is to sell all the produced products and satisfy the overall demand.

For simplicity, we assume that the firm cannot keep products as inventories from one period to the next and also it does not maximize its profits. This model does not take into account the financial constraints of the production process, and we assume that the firm has money to produce or to buy at any point in time. The main focus of the model is to show how the gap between the firm's expectations about the demand and the real demand in the market alters the price and what dynamics this process might produce. So the question is, how the firm knows if it had a successful sales campaign? We consider that a successful sales campaign means that all the products were sold. This is exactly the market equilibrium assumption except that in our model, it is just a temporal state of the system and not a constant reality of the market. The firm quantifies its success after each period using a very simple model - it divides the quantity demanded at time n by the quantity supplied at time n as shown in Eq. 5 . We call it the *signal of success* S,

$$S = \frac{D_n}{S_n}. \tag{5}$$

According to the signal of success, the firm decides how many goods to produce and supply in the next period of time. From the mathematical point of view, it is important to notice that the firm reacts to the signal of success and not implicitly to the quantities demanded and supplied. To introduce this idea in the model, we assume that the expected demand is proportional to the quantity supplied in the past sales season, but modulated by a function f that depends on the signal of success S,

$$D_{n+1}^{Exp} = f(S) \times S_n = D_n. \tag{6}$$

We call $f(S)$ the *multiplier of production*. This simple idea helps us to model the market assuming no inventories and inequalities between demand and supply. The signal of success can be divided in four subsets of outcomes, each one with its corresponding economic meaning. We assume that all outcomes are in the positive domain.

1. When $\frac{D_n}{S_n} = 0$, there is no demand, or even worst, there is no market. In this case, the firm will not produce anything for the next period due to the scarcity of demand.

2. When $0 < \frac{D_n}{S_n} < 1$, the quantity demanded is smaller than the quantity supplied at the given level of price. The firm manufactured more products than what the market could possibly absorb. From the economical point of view, the firm will probably confront economic losses and also gain negative expectations about the future state of the market.

3. When $\frac{D_n}{S_n} = 1$, the quantity demanded is exactly equal to the quantity supplied. This means that the firm had a successful sales campaign, exactly as we defined earlier. In general, firms aim to find themselves in this situation. This is a natural equilibrium point of the system as we will show in the following sections.

4. When $\frac{D_n}{S_n} > 1$, the quantity demanded is larger than the quantity supplied. This

is a stock-rupture situation. Although the firm has sold all the goods it produced, it is an unsatisfactory situation, since it is losing the possibility to sell even more goods and earn extra revenue. This contradicts Assumption 2. Imagine costumers entering through the shop door with money bills in their hands asking for some product that is out of stock. Although it has lost some extra revenue, the firm gains positive expectations about the future.

The model works as follow, in the first step the firm supplies some quantity of products to the market to get some feeling about the demand (seed). Then it observes the amount of demanded units at the price that it fixed. According to this quantity the firm decides how many units to produce or buy for the next period using a simple model to quantify the success of his sales campaign. We called it the signal of success, and it is a simple division between the demanded and supplied quantities at time n. After computing the signal of success the firm uses it to estimate the expected demand in the next period. The second step is the pricing process. The firm uses its ATC function to compute the average total cost of the products. After obtaining the cost per unit, it adds some profits over the cost using the gross margin operator. Finally, the firm introduces the goods with their new price into the market, it waits some time until it sees how many units have been sold, and then it repeats the whole process again.

3. Two types of firms and their behaviors

In this Section, we will describe two types of firm behaviors. A different function $f(S)$ is used to compute the multiplier of production for the next period of time in each case. This affects the amount of goods produced or bought in the present period of time for the coming sales season. In Fig. 2, we show this relationship. The expected demand is then computed according to Eq. 6. For the sake of simplicity, we have used two very simple firms that can be modeled analytically. But in the model, more complex firm behaviors could be introduced.

The naive supplier

The simplest assumption of all is that the firm makes the decision of how many goods to supply in the next period, using the signal of success and the amount of goods it supplied in the previous period as a bench mark. The firm uses a very simple model to compute the expected demand. It takes as the multiplier of production the signal of success itself. It multiplies then the signal of success with the quantity supplied in the previous period as shown in Eq. 8, to compute the expected demand,

$$f(S) = S = \tfrac{D_n}{S_n} \tag{7}$$

$$D_{n+1}^{Exp} = \left(\tfrac{D_n}{S_n}\right) \times S_n = D_n. \tag{8}$$

The logic behind this model is that the firm expects the demand to behave in the next sales season, exactly the same as it behaved in the previous period. This forecasting method is the same as the moving average method with exponential smoothing coefficient $\alpha = 1$, putting all the weight of the forecast on the most recent information [25]. There is a linear relationship between the signal of success and the multiplier as shown in Fig. 2. The firm is going to produce exactly the same quantity that was demanded in the previous period. For this reason, we have called naive supplier, to this firm. Considering all this, the model takes the following form

$$D_{n+1} = a - bP_{n+1}, \tag{9}$$

$$S_{n+1} = D_n, \tag{10}$$

$$P_{n+1} = \tfrac{1}{1-M} \cdot \big(\tfrac{F_c}{S_{n+1}} + v - vS_{n+1} + (S_{n+1})^2\big). \tag{11}$$

Simplifying this system of equations, we get the following onedimensional maps for the demand and the price,

$$D_{n+1}(1-M) = a(1-M) - b\big(\frac{F_c}{D_n} + v - vD_n + (D_n)^2\big), \tag{12}$$

$$P_{n+1} = \frac{1}{1-M} \cdot \Big(\frac{F_c}{a - b(P_n)} + v - v\big(a - b(P_n)\big) + \big(a - b(P_n)\big)^2\Big). \tag{13}$$

The cautious and optimistic supplier

This type of firm behavior is in fact a family of infinite number of behaviors, each one with a different sensitivity to the signal of success. This firm instead of merely using as multiplier of production the signal of success as it is, prefers to transform it to be able to improve the prediction of the demand in the next period. It uses a very simple but powerful model. It takes as the multiplier of production the nth root of the signal of success, where m defines his cautiousness and optimism as we will see next. The supplier multiplies the nth root of the signal of success with the quantity supplied in the previous period that serves it as bench mark. We can see this model in Eq. 15 ,

$$f(S) = \sqrt[m]{S} = \sqrt[m]{(\tfrac{D_n}{S_n})} \tag{14}$$

$$D_{n+1}^{Exp} = \sqrt[m]{(\tfrac{D_n}{S_n})} \times S_n, \tag{15}$$

where $m > 0$. From Fig. 2, we can see that when m increases the firm becomes less optimistic and more cautious about the future state of the market, when the signal of success is greater than one. But it becomes less cautious and more optimistic when the signal of success is between zero and one. This behavior is similar to what is described in the literature as loss aversion [26]. In this setup the firm takes as the reference point when the signal of success is equal to one. As the reader might guess the naive supplier is just a particular case in this model and it arises when $m = 1$.

So m determines the firm's sensitivity to the market states or to the signal of success perceived. In general, all of them behave in the same manner. When $\frac{D_n}{S_n} = 0$, and $\frac{D_n}{S_n} = 1$, there is no change in their behaviors, they expect the demand to be 0 and D_n, respectively, as we saw in the naive supplier case. The interesting behavior occurs when $0 < \frac{D_n}{S_n} < 1$, and when $\frac{D_n}{S_n} > 1$. In the first subset of outcomes, the firm perceives lower demand in proportion to the quantity supplied at time n. Because of that, it will produce fewer goods than before. Due to its optimism, it will produce a little bit more goods compared to what the naive supplier would had produced in the same situation. As m increases, the firm becomes more and more optimistic and it will produce more goods. On the other hand, when $\frac{D_n}{S_n} > 1$, the firm perceives high demand in proportion to the quantity supplied at time n. Therefore, it will produce more goods than before. However, its cautiousness will play an important role. It will produce fewer goods than the naive supplier in the same situation. As m increases, it is considered to be more cautious and it will produce less goods. We can write down this model as follows,

$$D_{n+1} = a - bP_{n+1}, \tag{16}$$

$$S_{n+1} = \sqrt[m]{(\frac{D_n}{S_n})} \times S_n, \tag{17}$$

$$P_{n+1} = \frac{1}{1-M} \cdot \big(\frac{F_c}{S_{n+1}} + v - vS_{n+1} + (S_{n+1})^2\big). \tag{18}$$

Simplifying this system of equations, we obtain the following twodimensional map for the demand and the supply,

$$D_{n+1}(1-M) = a(1-M) - b\big(\frac{F_c}{S_{n+1}} + v - vS_{n+1} + (S_{n+1})^2\big). \tag{19}$$

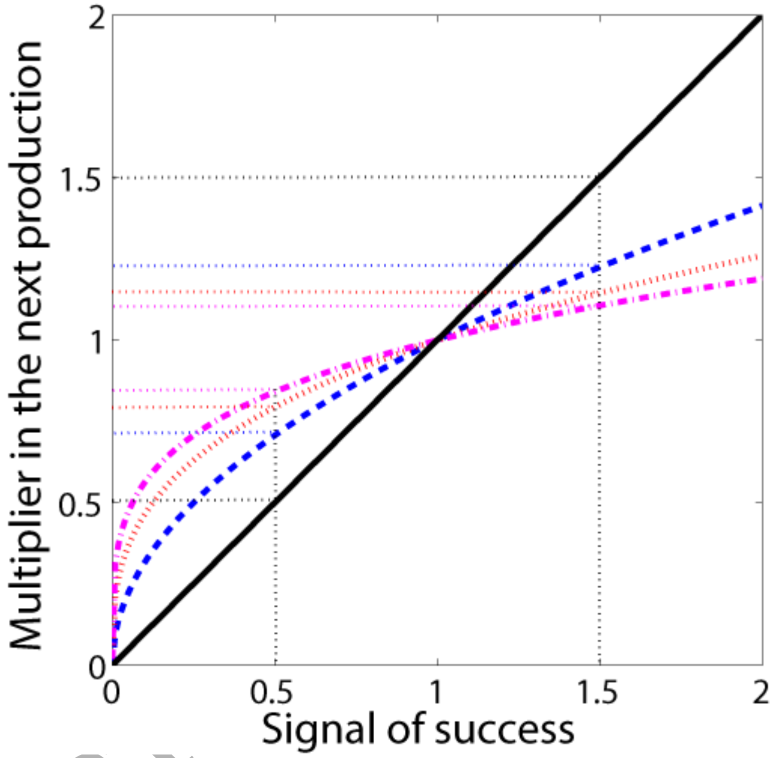

Figure 2: **Behaviors of suppliers in term of the signal of success.** The relationship between the nth root of the signal of success with the multiplier in the next production is shown in the figure above. The solid black curve represents the linear case or the naive supplier, $m = 1$. The blue dash line is the square root $m = 2$ of the signal of success. The red dot line is the cubic root $m = 3$ of the signal of success and the magenta dash-dot line is the $4th$ root of the signal of success. We have plotted the horizontal dot lines, to help the reader see the multiplier of production in each case, when the signal of success is 0.5 and 1.5.

$$S_{n+1} = \sqrt[m]{(\frac{D_n}{S_n})} \times S_n. \tag{20}$$

Here the firm needs two seeds to calculate the expected demand, D_0 and S_0. Notice that this two dimensional map can be reduced into a one dimensional map in terms of supply as shown in Eq. 21 .

$$S_{n+1} = \sqrt[2]{\frac{1}{S_n}\Big(\frac{1}{1-M}\big(a - b(\frac{F_c}{S_n} + v - vS_n + S_n^2)\big)\Big)} \times S_n. \tag{21}$$

4. Methodology

We have studied only two variations of the model. In Eq. 22, we show the naive supplier when the parameters are fixed as: $a = 10$, $b = 0.09$, $v = 4$, $F_c = 10$ and $M = 0.5$,

$$D_{n+1}(0.5) = 10 - 0.09\big(\frac{10}{D_n} + 4 - 4D_n + (D_n)^2\big). \tag{22}$$

In Eqs. 23 and 24, we show the cautious and optimistic supplier when the parameters are fixed as: $a = 30$, $b = 0.125$, $v = 6$, $F_c = 30$, $M = 0.5$ and $m = 2$.

$$D_{n+1}(0.5) = 15 - 0.125\big(\frac{30}{S_{n+1}} + 6 - 6S_{n+1} + (S_{n+1})^2\big), \tag{23}$$

$$S_{n+1} = \sqrt[2]{(\frac{D_n}{S_n})} \times S_n, \tag{24}$$

that can be easily simplified to the following one dimensional map,

$$S_{n+1} = \sqrt[2]{\frac{1}{S_n}\Big(2\big(30 - b(\frac{30}{S_n} + 6 - 6S_n + S_n^2)\big)\Big)} \times S_n. \tag{25}$$

We have studied the dynamics of both models using three tests. First, we have computed the time series of both models to observe the dynamics. We have changed the parameters b and M to see how the dynamics of the time series changes. We have decided to show only the chaotic time series because we want to prove the existence of chaos in the model. Secondly, we have plotted the bifurcation diagrams of the quantity demanded against the parameter b in both cases. We have done the same with the parameter M in the naive supplier case, to show the dynamics when the margin is changed. Lastly, we have computed the Lyapunov exponents spectrum of both systems.

5. Global dynamics and results

The naive supplier
In order to understand the relationship between the price and the quantity demanded, we have plotted the first 20 iterations of the model as shown in Fig. 3. We clearly see the price and the quantity demanded to behave exactly how we expected. When the prices are high, there is low demand and when the prices are low, there is high demand. However, the plots show an irregular behavior in both cases. The economical meaning of this behavior is that the firm and the consumers have not agreed on the quantity nor the price during the trade. In other words, their interactions were not translated into market equilibrium. Furthermore, it seems that this market is not efficient. But there is a small window between time steps 6 to 10, in which the trajectories of the price and the quantity demanded are almost flat or almost in equilibrium. However, after two time steps this behavior changes abruptly into high amplitude fluctuations. This is what is observed in real markets of ordinary goods. They behave dynamically and do not to fall into the frozen state that standard models predict. We did not obtain this behavior by an accident; we have chosen the parameter values precisely to get this behavior. Next, we will show that more dynamical behaviors are possible computing the bifurcation diagram.

For certain values of the parameters b, and M, it is possible to compute the fixed points of Eq. 13. If we allow the parameter b to vary between 0 and 0.0918, the equilibrium points of D_{n+1} are shown in the bifurcation diagram of D_{n+1} against b in Fig. 4. We also find a period-doubling route to chaos [27, 28] in Fig. 4, as b is increased beyond 0.0918. We have found period 6 and period 10 cycles when $b = 0.8531$ and $b = 0.0843999995$, respectively. We clearly see the huge range of demand dynamics when we are varying the parameter b. We will explain why this outcome is meaningful in terms of demand theory in the next Section. We obtain a similar bifurcation diagram when we vary M against D_{n+1}. In Fig. 6 we show how the quantity demanded is affected by the gross margin, when it is changed. Notice that in Fig. 6, $b = 0.03$. One can check in Fig. 4 that at this value of the parameter b, the system should be in equilibrium. Incrementing the gross margin in order to obtain more profits leads to a destabilization of the whole system. The model suggests that the firm greed has limits. This is the proof that the firm has influence on the global dynamics of the market. We have also computed the Lyapunov exponent spectrum to prove the existence of chaos in Fig. 5.

The cautious and optimistic supplier We start again with the time series as

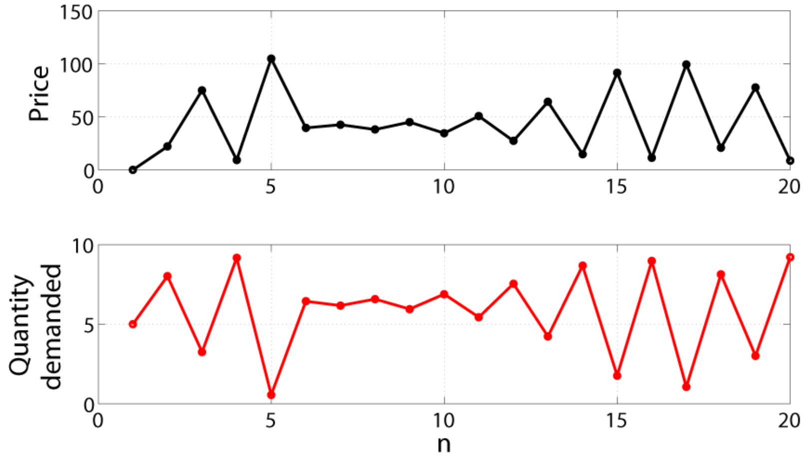

Figure 3: **The price and demand time series that correspond to the naive supplier during the first** 20 **periods of trade.** The two time series that are shown in the figure above were plotted iterating Eq. 12 and Eq. 21. The black line corresponds to the price, and the red line corresponds to the demanded quantity in the first 20 periods of trade. Despite the fact that the price and the demand are discrete quantities, it is easier to follow their evolution plotting them as continuous curves. But, note that the lines between the dots are meaningless.

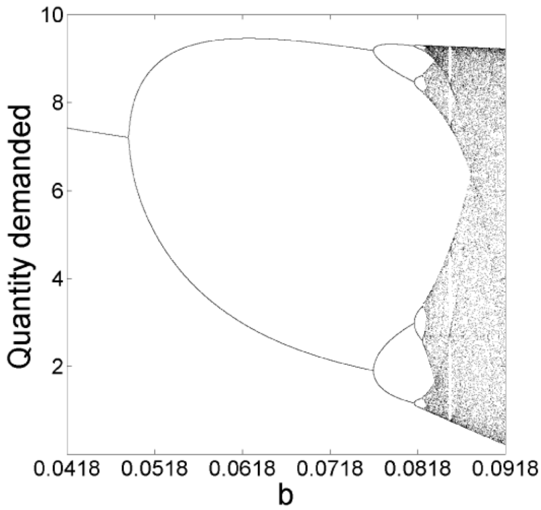

Figure 4: **The bifurcation diagram of the quantity demanded, D_{n+1}, against the parameter b.** We have divided the interval $(0.0418, 0.0918)$ of the parameter b into $10,000$ values. Then, we have set each value of parameter b in Eq. 12 and we have iterated the equation $3,000$ times until it settles down in the corresponding fixed points. Finally, we have plotted those fixed points against the value of parameter b to obtain this bifurcation diagram.

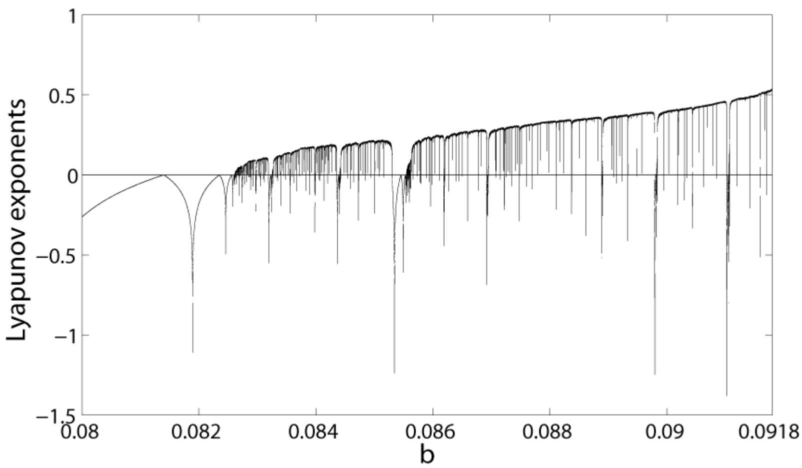

Figure 5: **The Lyapunov exponents spectrum corresponding to the naive supplier when parameter b is varied.** We have taken the interval $(0.08, 0.092)$ of the parameter b and we have computed the Lyapunov exponent of $100,000$ points within this interval. Finally, we have plotted the corresponding exponent against its corresponding value of parameter b to obtain the spectrum. The exponent is positive in a wide range of parameter b values, what proves the chaotic behavior of the system.

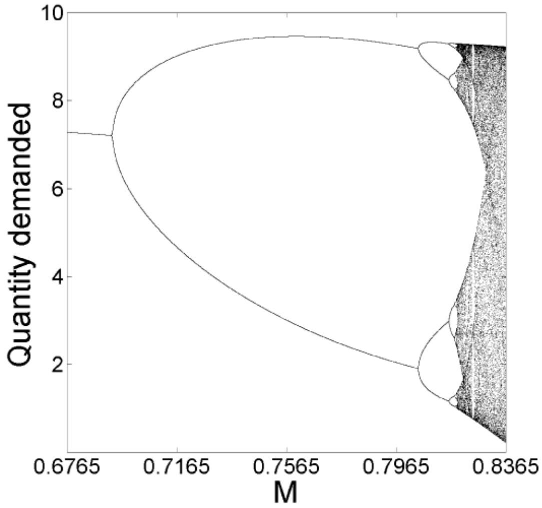

Figure 6: **The bifurcation diagram of the quantity demanded, D_{n+1}, against the parameter M.** We have divided the interval $(0.6765, 0.8365)$ of the parameter M into $20,000$ values. Then, we have set each value of parameter M in Eq. 12 and we have iterated the equation $3,000$ times until it settles down in the corresponding fixed points. Finally, we have plotted those fixed points against the value of parameter M to obtain this bifurcation diagram. Notice that when the gross margin is between 0 and 0.6765 the system is in equilibrium. This is a huge range of gross margin values. In contrast, only a small part of the gross margin interval causes the demand to behave chaotically. It is not a surprise that this small part corresponds to high margins.

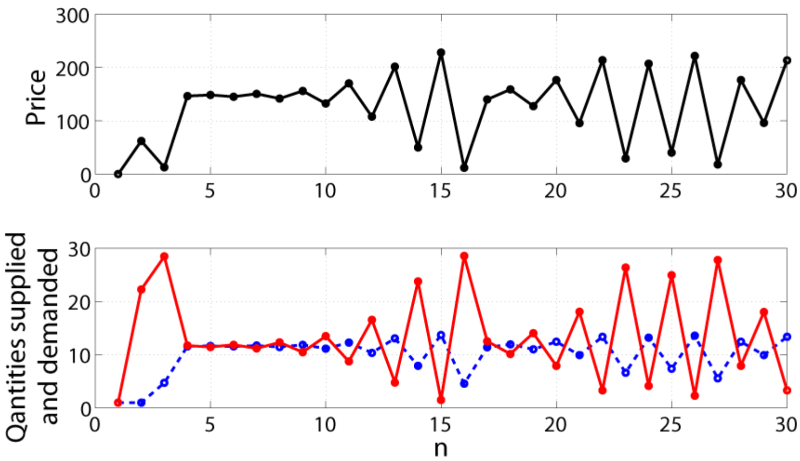

Figure 7: **Time series of the cautious and optimistic supplier in the first** 30 **periods of trade.** At the bottom we have plotted the demand D as a solid red line against the supply S as a dash blue line. Above in black, we have plotted the price trajectory in this trade scenario. This figure shows the dynamic behavior of the quantities supplied and demanded, and the price. The price is moving exactly as we would expect. There are periods where the price does not change much, so we can say the market is almost in equilibrium. And there are periods where the price changes dramatically, what corresponds to the nonequilibrium state of the market.

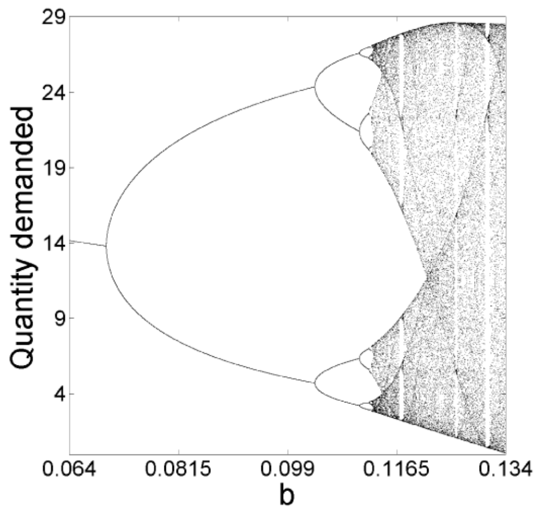

Figure 8: **The bifurcation diagram of the quantity demanded, D_{n+1}, against the parameter b.** We have divided the interval $(0.064, 0.134)$ of the parameter b into $10,000$ values. Then, we have set each value of parameter b in Eq. 19 and we have iterated the equation $3,000$ times until it settles down in the corresponding fixed points. Finally, we have plotted those fixed points against the value of parameter b to obtain this bifurcation diagram.

shown in Fig. 7. It is possible to verify how high prices have low demand and vice versa. We can see periods where the demanded and supplied quantities are almost the same. In these periods, the system is almost at equilibrium so the price is stable. But after some time, the system goes out of equilibrium and periodic-cycles and chaotic behavior arise. We have plotted the bifurcation diagram of D_{n+1} against b to illustrate some more possible behaviors as shown in Fig. 8. A period 3 orbit shows up when $b = 0.1308$. The existence of a period 3 orbit implies chaos [29]. We can see the period doubling route to chaos clearly in Fig. 8 as well. Furthermore, we have computed the Lyapunov exponent spectrum to prove the existence of chaos as shown in Fig. 9.

6. The final bifurcation means market collapse

In this Section, we will expand the economical assumptions of the model to emphasize the idea, that a final bifurcation can be a good description of a market collapse. We have chosen the naive supplier as a case study. But the reasoning

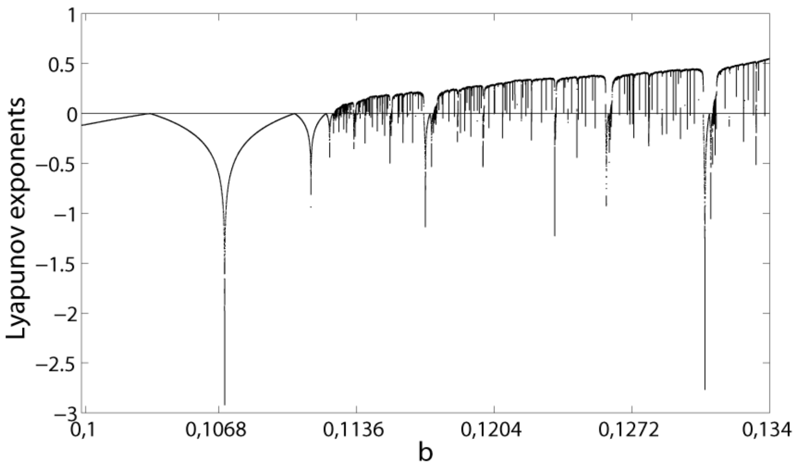

Figure 9: **The Lyapunov exponents spectrum corresponding to the cautious and optimistic supplier when parameter b is varied.** We have taken the interval $(0.1, 0.134)$ of the parameter b and we have computed the Lyapunov exponent of $80,000$ points within this interval. Finally, we have plotted the corresponding exponent against its corresponding value of parameter b to obtain the spectrum. The exponent is positive in a wide range of parameter b values, what proves the chaotic behavior of the system.

and the methodology that we have used to demonstrate this claim, is generic, and can be applied to all types of suppliers.

When the parameters are fixed in Eqs. 12 and 13 as: $D_1 = 1$, $S_1 = 1$, $a = 10$, $b = 0.095$, $v = 2$, $Fc = 20$ and $M = 0.5$, we get the following maps for the demand, the supply and the price :

$$D_{n+1}(0.5) = 10 - 0.095\big(\frac{20}{D_n} + 2 - 2D_n + (D_n)^2\big). \tag{26}$$

$$S_{n+1} = \sqrt[2]{\frac{1}{S_n}\Big(2\big(10 - 0.095(\frac{20}{S_n} + 2 - 2S_n + S_n^2)\big)\Big) \times S_n}. \tag{27}$$

$$P_{n+1} = \frac{\frac{20}{10-0.095(P_n)} + 2 - 2\big(10 - 0.095(P_n)\big) + \big(10 - 0.095(P_n)\big)^2}{1 - 0.5}. \tag{28}$$

Analyzing the time series produced by these maps, we find a transient chaotic behavior as shown in Fig. 10. The trajectories of the quantities demanded, the quantity supplied, and the price are completely chaotic until time step 69, where suddenly they explode. Beyond this point, the system starts to fluctuate without control giving arise quantities that are unscaled to the system or even infinitely large. We are not familiar with the complicated concepts of negative infinite price or infinite demand and supply. Therefore, to get a better economical understanding of this situation we need to extend our assumptions about the model.

We will first, focus on the demand side of the system. The meaning of parameter a in Eq. 1. is that when the product is freely available (its price is zero) in the market, the maximum amount of products that can be demanded is the value of parameter a. This is an accomplished fact, and it is the upper bound of units that can be demanded in this market, assuming the system lies in the positive domain. When we allowed the price to take negative values, the amount of products demanded was much higher from the value of parameter a. In this scenario the firm must pay the consumer to create the demand. We will assume that the firm does not make strategic decisions of this kind thinking on long time horizons. So, when the price is negative it just loses the incentives to supply. In Eq. 29, we include this new behavior into the model,

$$D_{n+1} = \begin{cases} 0 & \text{if } (bP_{n+1} > a), \\ \text{a} - bP_{n+1} & \text{if } (bP_{n+1} \leq a). \end{cases} \tag{29}$$

Following the same reasoning as in the demand case, we will extend our assumptions on the supply side of the system. The second assumption of the model is that the firm always tries to sell exactly the amount of goods it produced or bought. If it expects zero or negative demand, we can assume that the firm will not produce anything for the next period of time. It will probably get out of the market in this situation. The firm computes the expected demand before going into production, so if it sees that the expected demand is zero or negative, it stops immediately the process. We can describe mathematically this behavior using the following equations,

$$S_{n+1} = \begin{cases} \frac{1}{1-M} \cdot \left(\frac{F_c}{S_{n+1}} + v - vS_{n+1} + (S_{n+1})^2 \right) & \text{if } D_{n+1} > 0, \\ 0 & \text{if } D_{n+1}, \leq 0. \end{cases} \tag{30}$$

When the trajectories arrive to the final bifurcation, the market stops to exist immediately. The reader can see in Fig. 10, how after the final bifurcation the price stays at some high level where the quantities supplied and demanded go to zero. Note that if the demand crosses some critical value (small value), the system enters into a loop of destruction, due to the growing cost of production of diminishing quantities. We would expect similar dynamics in a situation of market collapse.

In real economies, we find two interesting properties that can be also observed in this model. The first one is the prediction problem, in which the collapse is impossible to forecast beforehand. Secondly, the global complexity of the market emerges from simple nonlinear interactions between the economical agents.

7. The influence of the price elasticity of demand (PED) on the global dynamics

We have modeled the demand as a monotonic function. Nevertheless, the slope of the demand curve, parameter b, has a huge effect on the dynamics of the system as we saw in the previous sections. To capture this idea we can compute the *price elasticity of demand* (PED), which measures the sensitivity to the price of the quantity demanded, and it is given by the following ratio:

$$PED = \frac{dD/D}{dP/P}. \tag{31}$$

In our model, given that we have a linear relationship between price and demand described by Eq. 1, the PED is by,

$$PED = -b \left(\frac{P}{a - bP} \right). \tag{32}$$

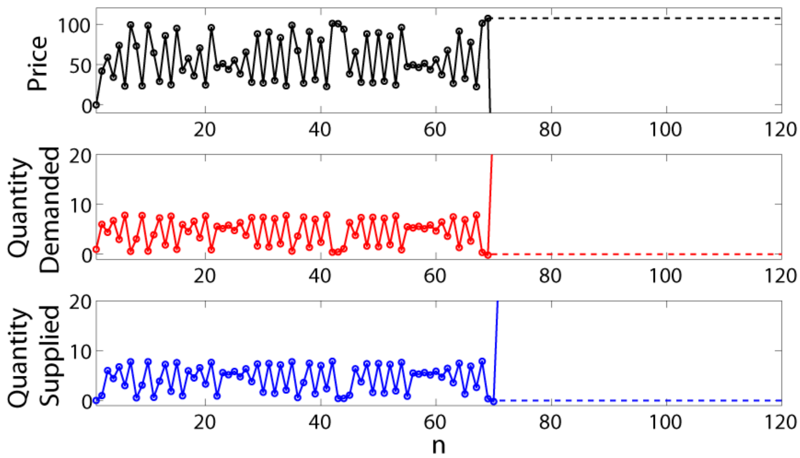

Figure 10: **Time series of the quantities demanded and supplied before and after bounding the system.** The solid line represents the time series of the price, the demand and the supply, simply by iterating the maps fixing the parameters as: $D_1 = 1$, $a = 10$, $b = 0.095$, $v = 2$, $Fc = 20$ and $M = 0.5$. The time series behave chaotically until time step 69 where a very big fluctuation occurs. The price become negative so the quantities demanded and supplied increase dramatically. The dash line represent the same system as before but now bounded. The time series can not be negative so that, when some critical value is crossed the system simply goes to zero, as in the case of the quantity demanded and supplied shown in the figure above.

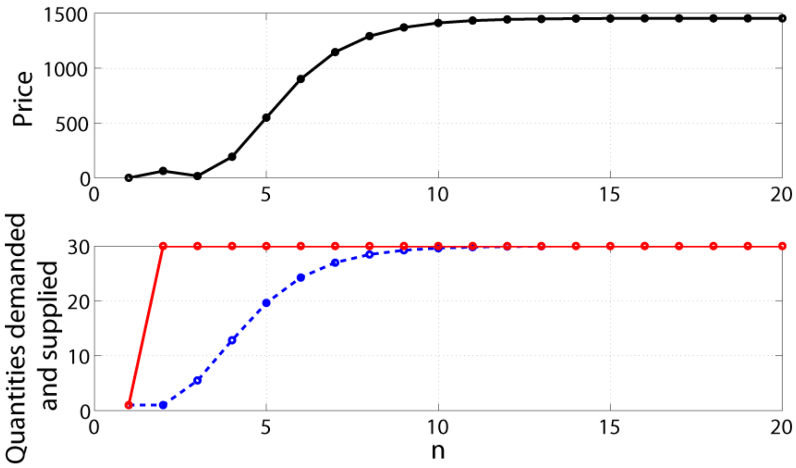

Figure 11: **Dynamics of supply when there is a perfectly elastic demand curve.** Time series of the first 20 periods of trade in the cautious and optimistic supplier case when $b = 0$. At the bottom we have plotted the demand D in red against the supply S in blue. Above we have plotted the price trajectory of this trade scenario.

In general, products which are elastic tend to have many substitutes, they must be bought frequently and they are traded in very competitive markets. In this model we have assumed all of the above. We have done this by modeling the market as an ordinary goods market that obeys the demand law.

When we vary the parameter b, we change the price elasticity of demand. For example, when $b = 0$, we encounter a perfectly elastic demand curve. One can imagine the demand curve as an horizontal line. In Fig. 11, we clearly see how the quantity supplied in blue is rapidly sticking to the quantity demanded in red until all the demand is fulfilled. Due to the excess demand, the price is going up until it reaches the market equilibrium price. This process is not instantaneous, as can be checked. Even though we have assumed a perfectly elastic demand, the firm does not know it. It takes it about 13 periods of trade to supply all the goods demanded by the market. This is a good example of the adjustment dynamics that underlies the market equilibrium assumption.

But the really remarkable result is that a very small change in the PED can change completely the system dynamics. In Fig. 8 we describe how the global dynamics of the system changes as we increase the value of the parameter b inside a very small subset. When $0 < b < 0.134$, we observe equilibrium points, cycles and chaotic trajectories, but when $b > 0.134$, the system blows up. We have showed in the previous Section that the economical meaning of this is a market collapse.

This behavior is not special only for b, when the value of M and a are varied, we encounter the same dynamics, but we assume that the gross margin value is controlled by the firm. Therefore, theoretically the firm can avoid erratic trajectories or crash scenarios manipulating this variable. We have focused on the price elasticity of demand because it cannot be influenced by the firm but it is directly related to the price. Exactly like in the real world, the small firm tries to adjust its production to the demand, and not the demand to the production. Because trying to influence the demand is highly expensive and only big companies with more resources can afford it.

8. Conclusions

We have introduced the supply based on demand model studying two types of firm behaviors, the naive supplier and the cautious and optimistic supplier. In both cases, we have found that the model is capable of reproducing a large variety of dynamics such as equilibrium, cycles, chaos, and even catastrophic dynamics under simple and reasonable economic assumptions. We have emphasized the idea that the final bifurcation can be a good description of a market collapse by adding some new assumptions to the model. We have shown the important role that the price elasticity of demand plays on the global dynamics of the market. One important result is that very small changes in the price elasticity of demand leads to very different global dynamics assuming a monotonic demand function. We have also demonstrated the huge influence of the gross margin, M, on the market dynamics.

Acknowledgments

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